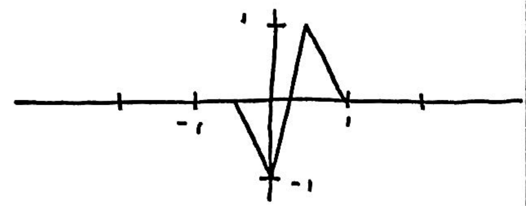
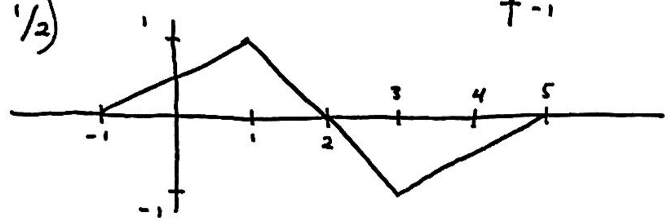
Solutions - Homework 1

1) Draw the following signals:

a) $x(2(-t+\frac{1}{2})) = x(-2t+1)$.



b) $x\left(\frac{t-1}{2}\right) = x\left(\frac{1}{2}t - \frac{1}{2}\right)$

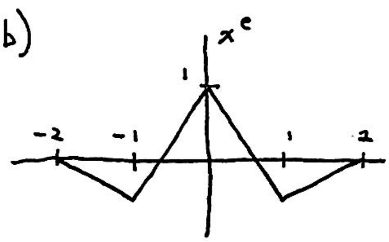


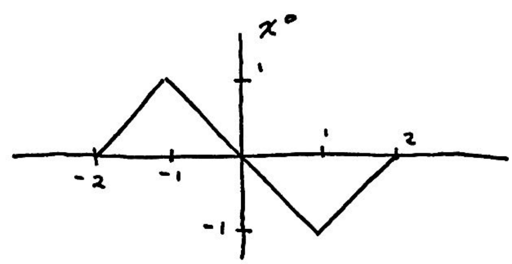
2) Express Π as a simple combination of unit step functions:

$$\Pi(x) = u(x+\tfrac{1}{2}) - u(x-\tfrac{1}{2}).$$

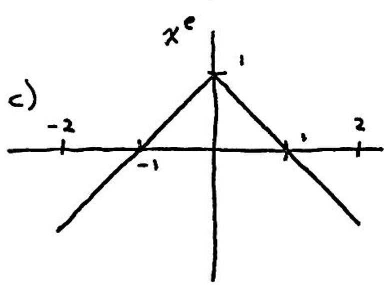
3) a) $f^e = \frac{1}{2}\left(f(x) + f(-x)\right) \qquad f^o(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$.

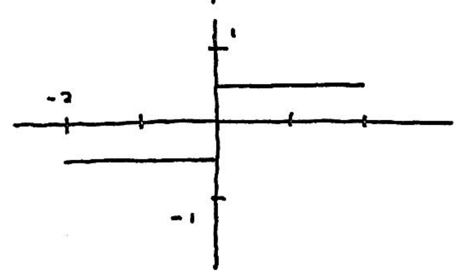
b)





c)





4) $\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) \qquad \sin(\omega t) = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$.

6) a) $f'(x) = \exp(i8x)$

b) $\int f(x)\,dx = \frac{9i}{\pi} \exp(-i2\pi x) + C$

$f'(x) = -i\,36\pi \exp(-i2\pi x)$

7) Find the value of $\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx$.

Let $I = \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx$. $\quad I^2 = \left(\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-\pi y^2}\,dy\right)$.

$\Rightarrow I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)}\,dx\,dy.$

Changing to polar coordinates,

$$I^2 = \int_0^{2\pi}\int_0^{r} e^{-\pi r^2}\, r\,dr\,d\theta.$$

Let $u(r) = r^2$. Then $u'(r) = 2r\,dr$.

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\pi u(r)}\,(1/2)\,u'(r)\,dr\,d\theta$$

$$= \pi\int_0^{\infty} e^{-\pi u(r)}\,u'(r)\,dr.$$

Note that $\frac{d}{dr} e^{-\pi u(r)} = e^{-\pi u(r)}\left(-\pi\,u'(r)\right)$.

$\Rightarrow I^2 = -e^{-\pi u(r)}\Big|_0^{\infty} = 1. \qquad \therefore I^2 = \pm 1.$

Since $I = \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx$ is an integration of positive values, I must be positive.

$\boxed{\therefore I = 1.}$

8) There are two square roots of x:

$$\sqrt{|x|}\,\underline{\angle\, \tfrac{1}{2}\theta_x} \quad\text{and}\quad \sqrt{|x|}\,\underline{\angle\, \tfrac{1}{2}\theta_x + \pi}\,.$$

The n n^{th} roots of x are

$$\sqrt[n]{|x|}\,\underline{\angle\, \tfrac{1}{n}\theta_x},\ \sqrt[n]{|x|}\,\underline{\angle\, \tfrac{1}{n}\theta_x + \pi/n},\ \ldots$$

$$\sqrt[n]{|x|}\,\underline{\angle\, \tfrac{1}{n}\theta_x + \tfrac{(n-1)\pi}{n}}$$

9) a) Since x is periodic, $x(3T) = x(0)$.
Since x is odd, $x(t) = -x(-t)$ for all t.
$\Rightarrow x(0) = -x(0)$. $\therefore x(0) = \underline{0 = x(3T)}$.

b) T_1/T_2 must be a rational number

c) The function is constant

10) a) $x_1 x_2 = \sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}\ \underline{/\arctan(a_1,b_1)+\arctan(a_2,b_2)}$

b) $x_1/x_2 = \dfrac{\sqrt{a_1^2+b_1^2}}{\sqrt{a_2^2+b_2^2}}\ \underline{/\arctan(a_1,b_1)-\arctan(a_2,b_2)}$

11) a) $0+0=0$. For any $a \in \mathbb{R}$, $a\cdot(0+0)=a\cdot 0$.

$\Rightarrow a\cdot 0 + a\cdot 0 = a\cdot 0$

$\Rightarrow [a\cdot 0 + a\cdot 0] + {}^-(a\cdot 0) = a\cdot 0 + {}^-(a\cdot 0)$

$\Rightarrow a\cdot 0 + [a\cdot 0 + {}^-(a\cdot 0)] = 0$

$\Rightarrow a\cdot 0 + 0 = 0 \Rightarrow a\cdot 0 = 0$.

b) Let $a \in \mathbb{C}$. $a\cdot 0 = (|a|\,\underline{/\theta_a})(0\,\underline{/0}) = 0\,\underline{/\theta_a} = 0$.

12) It's periodic.

13) Evaluate $\frac{1}{P}\int_{p_0}^{p_0+P} \exp\left(i2\pi\frac{nx}{P}\right)\exp\left(-i2\pi\frac{mx}{P}\right)dx$.

I will denote this value as V.

If $n=m$, $V = \frac{1}{P}\int_{p_0}^{p_0+P} dx = 1$.

If $n\neq m$,

$$V = \frac{1}{P}\int_{p_0}^{p_0+P}\exp\left(i2\pi\frac{(n-m)}{P}x\right)dx$$

$$= \frac{1}{P}\frac{1}{i2\pi\frac{(n-m)}{P}}\left[\exp\left(i2\pi\frac{(n-m)}{P}(p_0+P)\right)-\exp\left(i2\pi\frac{(n-m)}{P}p_0\right)\right]$$

$$= \frac{1}{P}\frac{1}{i2\pi\frac{(n-m)}{P}}\exp\left(i2\pi\frac{(n-m)}{P}p_0\right)\left[\exp\left(i2\pi\frac{(n-m)}{P}P\right)-1\right].$$

$\exp(i2\pi(n-m)) = \cos(2\pi(n-m)) + i\sin(2\pi(n-m)) = 1$.

$\Rightarrow V=0$ when $n\neq m$.

$\therefore V = \begin{cases}1 & \text{if } n=m\\ 0 & \text{otherwise}\end{cases}$.

15) a) $P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(\gamma)|^2 d\gamma = \lim_{T\to\infty} \frac{1}{2T} E_x$.

Since E_x is finite, $P_x = 0$.

b) If $|E_x| < \infty$ then $P_x = \lim_{T\to\infty} \frac{1}{2T} E_x = 0$.

But this is a contradiction since, by definition, $P_x > 0$.

$\therefore E_x = \infty$ (meaning that the integral diverges).