Assignment 2 - Solutions

1) Given two points (x_1, y_1) and (x_2, y_2), the linear interpolation between them is (x, y) where

$$y = y_1 + (x - x_1)\frac{y_1 - y_0}{x_1 - x_0}.$$

2) Dot product is linear in the first argument. It is conjugate linear (but not linear) in the second argument.

3) $x_1(t) = \cos(t) + t^2$. $\quad x_2(t) = -t^2$.
Neither x_1 nor x_2 are periodic but $x_1(t) + x_2(t) = \cos(t)$ is periodic.
So x_1 and x_2 need not be periodic in order for $x_1 + x_2$ to be periodic.

4) $x(\gamma) = (1/\gamma)\, u(\gamma - 1)$.

$$E_x = \int_{-\infty}^{\infty} |x(\gamma)|^2 \, d\gamma = \int_{-\infty}^{\infty} (1/\gamma)^2 u(\gamma-1)\, d\gamma$$
$$= \int_1^{\infty} \frac{1}{\gamma^2}\, d\gamma = -1/\gamma \Big|_1^{\infty} = 1.$$

$\therefore$ x is an Energy function.

$$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(\gamma)|^2 \, d\gamma = \lim_{T\to\infty} \frac{1}{2T} E_x = 0.$$

$\therefore$ x is not a power function.

5) a) $S\{x\}(\gamma) = x(\gamma)\sin(\omega\gamma + \phi)$.

Claim: S is linear.
Proof:
$$S\{x_1 + x_2\}(\gamma) = (x_1 + x_2)(\gamma)\sin(\omega\gamma + \phi)$$
$$= x_1(\gamma)\sin(\omega\gamma+\phi) + x_2(\gamma)\sin(\omega\gamma+\phi)$$
$$= S\{x_1\}(\gamma) + S\{x_2\}(\gamma).$$

$$S\{\alpha x_1\} = \alpha x_1(\gamma)\sin(\omega\gamma+\phi)$$
$$= \alpha[x_1(\gamma)\sin(\omega\gamma+\phi)] = \alpha S\{x\}(\gamma).$$ ∎

Claim: ~~Let $g(t) = x(t-\gamma)$.~~
S is time variant.
Proof: Let $g(t) = x(t-\Delta)$. $S\{g\}(\gamma) = g(\gamma)\sin(\omega\gamma+\phi)$.

$$\Rightarrow S\{x(t-\Delta)\}(\gamma) = x(\gamma-\Delta)\sin(\omega\gamma+\phi) \neq S\{x\}(\gamma-\Delta).$$ ∎

S is causal and memoryless.

b) $S\{x\}(t) = t\, x'(t)$.

Claim: S is linear.
Proof: $S\{x_1 + x_2\}(t) = t\,(x_1 + x_2)'(t) = t\,[x_1'(t) + x_2'(t)]$
$= t\,x_1'(t) + t\,x_2'(t) = S\{x_1\}(t) + S\{x_2\}(t)$.

$S\{\alpha x\}(t) = t\,(\alpha x)'(t) = \alpha\, t\, x'(t) = \alpha\, S\{x\}(t)$. ∎

Claim: S is time variant.
Proof: ~~$S\{x(\gamma$~~
Let $g(\gamma) = x(\gamma - \Delta)$.
$S\{x(\gamma - \Delta)\}(t) = t\,g'(t) = t\,x'(t-\Delta) \neq (t-\Delta)\,x'(t-\Delta)$
$= S\{x(\gamma)\}(t-\Delta)$.

S is causal and has memory.

c) $S\{x\}(\gamma) = 1 + x(\gamma)\cos(\omega\gamma)$.

Claim: S is not linear.
Proof: $S\{0\} = 1 \neq 0$.

Claim: S is shift variant.
Proof: Let $g(t) = x(t-\Delta)$.
$S\{g\}(\gamma) = 1 + g(\gamma)\cos(\omega\gamma) = 1 + x(\gamma-\Delta)\cos(\omega\gamma)$
$\neq 1 + x(\gamma-\Delta)\cos(\omega(\gamma-\Delta)) = S\{x\}(\gamma-\Delta)$.

S is causal and memoryless.

d) $S\{y\}(t) = \cos(\omega t + y(t))$.

Claim: S is not linear.
Proof: $S\{0\}(t) = \cos(\omega t) \neq 0$.

Claim: S is shift variant.
Proof: Let $g(\gamma) = y(\gamma - \Delta)$.
$S\{g\}(t) = \cos(\omega t + g(t)) = \cos(\omega t + y(t-\Delta))$.
$\neq \cos(\omega(t-\Delta) + y(t-\Delta))$
$= S\{y\}(t-\Delta)$.

S is memoryless and causal.

Non-causal and has memory.
To see that it's non-causal, consider any x<0.

e) $S\{y\}(x) = \int_{-x}^{x} y(\tau)\,d\tau$.

Claim: S is linear.

Proof:

$S\{y_1 + y_2\}(x) = \int_{-x}^{x} (y_1 + y_2)(\tau)\,d\tau$

$= \int_{-x}^{x} y_1(\tau)\,d\tau + \int_{-x}^{x} y_2(\tau)\,d\tau$

$= S\{y_1\}(x) + S\{y_2\}(x)$.

$S\{\alpha y\}(x) = \int_{-x}^{x} (\alpha y)(\tau)\,d\tau = \alpha \int_{-x}^{x} y(\tau)\,d\tau$

$= \alpha\, S\{y\}(x)$. ∎

Claim: S is shift variant.

Proof:

Let $y = u$ (the Heaviside step function).

$S\{u\}(x) = \int_{-x}^{x} u(\tau)\,d\tau = x$.

$S\{u(\gamma - \Delta)\}(x) = \int_{-x}^{x} u(\tau - \Delta)\,d\tau$

$= \max(\min(x + \Delta, 2x), 0)$

$\neq x - \Delta = S\{u\}(x - \Delta)$. ∎

f) $S\{z\}(\gamma) = \int_{-\infty}^{\gamma/2} z(\tau)\,d\tau$.

Claim: S is linear.

Proof: $S\{z_1 + z_2\}(\gamma) = \int_{-\infty}^{\gamma/2} (z_1 + z_2)(\tau)\,d\tau = \int_{-\infty}^{\gamma/2} z_1(\tau)\,d\tau + \int_{-\infty}^{\gamma/2} z_2(\tau)\,d\tau$

$= S\{z_1\}(\gamma) + S\{z_2\}(\gamma)$. ∎

Claim: S is shift variant.

Proof: $S\{z(t-k)\}(\gamma) = \int_{-\infty}^{\gamma/2} z(\tau - k)\,d\tau$.

Let $u = \tau - k$. When $\tau = \gamma/2$, $u = \gamma/2 - k$.

$\Rightarrow S\{z(t-k)\}(\gamma) = \int_{-\infty}^{\frac{\gamma}{2}-k} z(u)\,du = S\{z\}(\gamma - 2k) \neq S\{z\}(\gamma - k)$. ∎

S has memory.
Considering the value of $S\{z\}(\gamma)$ when $\gamma = -4$, we see that S is non-causal.

g) $S\{y(t+3)\}(\gamma) = y(\gamma+2) + y(\gamma-3)$.

Let $g(t) = y(t+3) \Rightarrow y(\gamma+2) = g(\gamma-1)$ and $y(\gamma-3) = g(\gamma-6)$.

$\Rightarrow S\{g(t)\}(\gamma) = g(\gamma-1) + g(\gamma-6)$.

S is linear and shift invariant.
S is causal and has memory.

6) a) $x(t) = 4\Lambda(t/2) - 6\Lambda(t) + 4\Lambda(2t)$.

b) $x(t) = 2\Lambda(t/2) - 2\Lambda(t) + \Pi(t/2)$.

7) $$S\{x\}(t) = \frac{x^2(t)}{\int_{-\infty}^{t} x(\tau)\,d\tau}.$$

$S\{0\}(t) = 0/0$, which is not defined.

$\therefore$ S is not linear.

8) $G\{x\} = x^e$.

Claim: G is linear.
Proof:
$G\{x_1 + x_2\} = (x_1 + x_2)^e = x_1^e + x_2^e$.

$G\{\alpha x\} = (\alpha x)^e = \alpha x^e$.

Claim: G is shift variant.
Proof:
Let $x(t) = u(t)$. $\Rightarrow G\{x\} = 1$.

$G\{x(t-1)\}(t) = u(-t+1) + u(t-1)$.

9) Suppose x is periodic with period T.

Let $g(t) = x(t-T)$. Since x is periodic, $g = x$.

$S\{g(t)\}(\gamma) = S\{x(t-T)\}(\gamma) = S\{x\}(\gamma - T) = y(\gamma - T)$
since S is shift invariant.

$S\{g(t)\}(\gamma) = S\{x(t)\}(\gamma) = y(\gamma)$ since $g = x$.

$\therefore$ the output is periodic with the same period as the input.

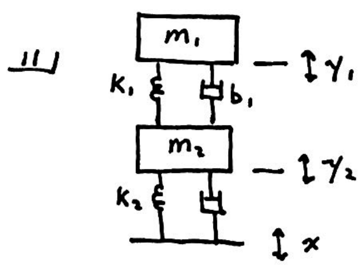
10) Claim: $x \cdot y = \overline{y \cdot x}$.

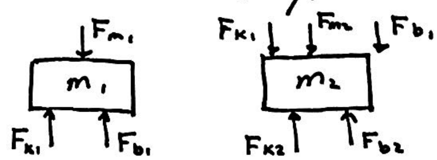
Proof:

$x \cdot y = x_1 \overline{y}_1 + x_2 \overline{y}_2 + \cdots + x_N \overline{y}_N$.

$\overline{y \cdot x} = \overline{y_1 \overline{x}_1 + y_2 \overline{x}_2 + \cdots + y_N \overline{x}_N} = \overline{y}_1 x_1 + \overline{y}_2 x_2 + \cdots + \overline{y}_N x_N$
$= x \cdot y$.

11)





2 Equations:

$F_{m_1} = F_{k_1} + F_{b_1}$ ①

$F_{m_2} = F_{k_2} + F_{b_2} - F_{k_1} - F_{b_1}$. ②

① $-m_1 y_1'' = k_1(y_2 - y_1) + b_1(y_2 - y_1)'$

② $-m_2 y_2'' = k_2(x - y_2) + b_2(x - y_2)' - k_1(y_2 - y_1) - b_1(y_2 - y_1)'$.

The input is x and the output is y_1. $S\{x\} = y_1$.

Claim: S is linear.

To prove S is linear, we will consider the augmented system $\hat{S}\{x\} = (y_2, y_1)$.

If $\hat{S}$ is linear then S is linear.

Claim: $\hat{S}$ is linear.

To show $\hat{S}$ is linear, we must show two things are true:

1) If (x_a, y_{1a}, y_{2a}) and (x_b, y_{1b}, y_{2b}) are solutions then $(x_a + x_b, y_{1a} + y_{1b}, y_{2a} + y_{2b})$ is a solution.
2) If (x, y_1, y_2) is a solution then $(\alpha x, \alpha y_1, \alpha y_2)$ is a solution.

To show (1):

$$-m_1(y_{1a}+y_{1b})'' = k_1\left((y_{2a}+y_{2b})-(y_{1a}+y_{1b})\right) + b_1\left((y_{2a}+y_{2b})-(y_{1a}+y_{1b})\right)'$$

$$\Leftrightarrow -m_1 y_{1a} - k_1(y_{2a}-y_{1a}) - b_1(y_{2a}-y_{1a})' = +m_1 y_{1b} + k_1(y_{2b}-y_{1b}) + b_1(y_{2b}-y_{1b})'.$$

Both sides of this equation are 0 since we assumed (x_{1a}, y_{1a}, y_{2a}) and (x_{1b}, y_{1b}, y_{2b}) are solutions.

Need to do the same for equation ②.

To show (2):

$$① \; -m_1(\alpha y_1)'' = k_1(\alpha y_2 - \alpha y_1) + b_1(\alpha y_2 - \alpha y_1)'$$

$$\Leftrightarrow \alpha(-m_1 y_1)'' = \alpha\left[k_1(y_2-y_1) + b_1(y_2-y_1)'\right]$$

which is true. Divide both sides by α to realize the equation is solved by our assumption.

Need to show the same for ②.

By the above, $\tilde{S}$ is linear. Since $\tilde{S}$ is linear, S is linear. ∎

To show the system is shift invariant, we need to show the following:

If $(x(t), y_1(t))$ is a solution then $(x(t-\Delta), y_1(t-\Delta))$ is a solution $\forall\ \Delta$.

We will again use the augmented system and show that if $(x(t), y_1(t), y_2(t))$ is a solution of $\tilde{S}$ then $(x(t-\Delta), y_1(t-\Delta), y_2(t-\Delta))$ is a solution of $\tilde{S}$.

For equation ①:

$$-m_1 y_1''(t-\Delta) = k_1\left(y_2(t-\Delta) - y_1(t-\Delta)\right) + b_1\left(y_2(t-\Delta) - y_1(t-\Delta)\right)'$$

$$\Leftrightarrow -m_1 y_1''(u) = k_1\left(y_2(u) - y_1(u)\right) + b_1\left(y_2(u) - y_1(u)\right)'$$

where $u = t-\Delta$.

We know this is true by assumption.

Need to show the same for ②.

12/ $y''(x) - 36y'(x) + 224y(x) = 0.$

$s^2 - 36s + 224 = 0 \Rightarrow (s-28)(s-8) = 0 \Rightarrow s \in \{8, 28\}.$

$\Rightarrow$ y has the form $y(x) = C_1 e^{28x} + C_2 e^{8x} = 0.$

From the initial condition, $C_1 + C_2 = 0 \Rightarrow C_1 = -C_2.$

$\therefore$ y has the form $y(x) = Ce^{28x} - Ce^{8x} = 0,$

where $C \in \mathbb{R}.$