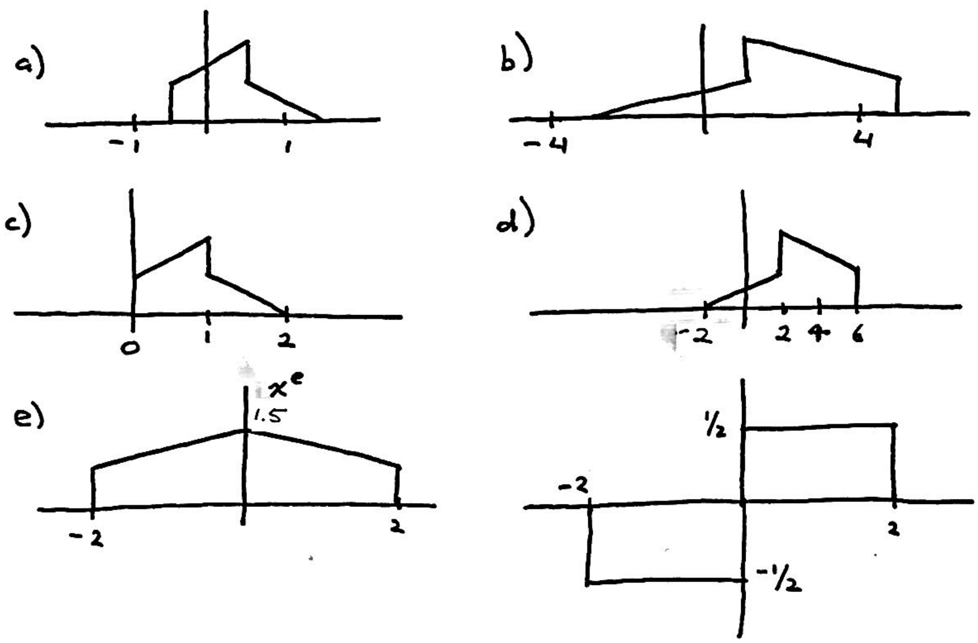
Assignment 3 - Solutions

1)



2) Let $u = s - \Delta$. $\Rightarrow \int_{-\infty}^{\infty} x(s)\,\delta(s-\Delta)\,ds = \int_{-\infty}^{\infty} x(u+\Delta)\,\delta(u)\,du$

$$= x(\Delta).$$

b) Case 1: $a > 0$.

Let $u = as \Rightarrow du = a\,ds$. $\Rightarrow \int_{-\infty}^{\infty} \delta(as)\,ds = \int_{-\infty}^{\infty} \delta(u)\,\frac{1}{a}\,du.$

$\therefore\ \delta(as) = \frac{1}{a}\,\delta(s).$

Case 2: $a < 0$.

Let $u = as \Rightarrow du = a\,ds$. $\Rightarrow \int_{-\infty}^{\infty} \delta(as)\,ds = \int_{\infty}^{-\infty} \delta(u)\,\frac{1}{a}\,du$

$$= \int_{-\infty}^{\infty} \delta(u)\,\frac{1}{|a|}\,du.$$

$\therefore\ \delta(as) = \frac{1}{|a|}\,\delta(s).$

In all cases, $\underline{\delta(as) = \frac{1}{|a|}\,\delta(s)}$.

3/ a) Show $f*(g*h) = (f*g)*h$.

Let $v(t) = (g*h)(t) = \int_{-\infty}^{\infty} g(\tau)\,h(t-\tau)\,d\tau$.

$$(f*v)(t) = \int_{-\infty}^{\infty} f(\gamma)\,v(t-\gamma)\,d\gamma = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\gamma)\,g(\tau)\,h(t-\gamma-\tau)\,d\gamma\,d\tau.$$

Let $z(t) = (f*g)(t) = \int_{-\infty}^{\infty} f(\tau)\,g(t-\tau)\,d\tau$.

$$(z*h)(t) = \int_{-\infty}^{\infty} z(\gamma)\,h(t-\gamma)\,d\gamma = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)\,g(\gamma-\tau)\,h(t-\gamma)\,d\tau\,d\gamma.$$

Let $s = \gamma - \tau \Rightarrow \gamma = s + \tau$.

$$\Rightarrow ((f*g)*h)(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)\,g(s)\,h(t-s-\tau)\,ds\,d\tau$$

$$= (f*(g*h))(t).$$ ■

b) Show $f*g = g*h$.

$(f*g)(t) = \int_{-\infty}^{\infty} f(\tau)\,g(t-\tau)\,d\tau$. Let $u = t-\tau$.

$$\Rightarrow (f*g)(t) = \int_{-\infty}^{\infty} f(t-u)\,g(u)\,du = (g*f)(t).$$ ■

c) Show $f*(g+h) = (f*g) + (f*h)$.

$$(f*(g+h))(t) = \int_{-\infty}^{\infty} (g+h)(\tau)\,f(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} g(\tau)\,f(t-\tau)\,d\tau + \int_{-\infty}^{\infty} h(\tau)\,f(t-\tau)\,d\tau$$

$$= (f*g)(t) + (f*h)(t).$$

d) Show $f*\delta = f$.

$(f*\delta)(t) = \int_{-\infty}^{\infty} \delta(\tau)\,f(t-\tau)\,d\tau = f(t)$ by the sifting property.

4/ a) $y = x * u$ where u is the step function.

b) $y(s) = (x(s) * u(s)) - (x(s) * u(s-T))$

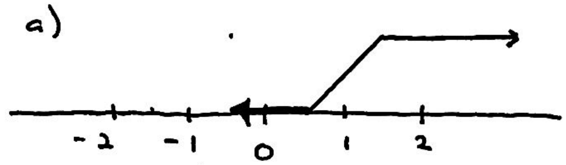
$$= x(s) * \Pi\left(\frac{s}{T} - \frac{T}{2}\right).$$

c) $y = x * \delta$.

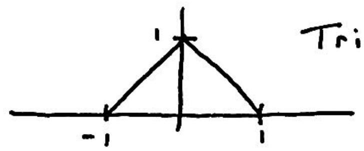
d) $y(\nu) = x(\nu) * \delta(\nu - 1)$.

e) $y(\nu) = x(\nu) * \delta(\nu + 1)$.

5/ a)



b)



Tri function

6/ a) $f(x) = \int_{-\infty}^{x} \delta(\gamma)\, d\gamma = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$.

This is the step function.

b) $\Pi(x) = u(x + 1/2) - u(x - 1/2)$

$$= \int_{-\infty}^{x+1/2} \delta(\gamma)\, d\gamma - \int_{-\infty}^{x-1/2} \delta(\gamma)\, d\gamma.$$

$\Rightarrow \frac{d}{dx}\Pi(x) = \delta(x + \frac{1}{2}) - \delta(x - \frac{1}{2})$ by the Fundamental Theorem of Calculus.

7/ a) Using integration by parts

$$\int_{-\infty}^{\infty} \delta'(s) f(s)\, ds = \delta(s) f(s)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(s) f'(s)\, ds.$$

$\delta(s) f(s)\Big|_{-\infty}^{\infty} = 0$ since $\delta(s) = 0 \;\forall\; s \neq 0$.

$$\Rightarrow \int_{-\infty}^{\infty} \delta'(s) f(s)\, ds = -\int_{-\infty}^{\infty} \delta(s) f'(s)\, ds = -f'(0)$$

by the sifting property.

Claim: $\int_{-\infty}^{\infty} \delta^{(K)}(s)\, f(s)\, ds = (-1)^K f^{(K)}(0)$.

Proof: Using integration by parts

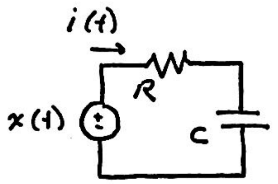
$$\int_{-\infty}^{\infty} \delta^{(K)}(s)\, f(s)\, ds = \delta^{(K-1)}(s)\, f(s) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta^{(K-1)}(s)\, f'(s)\, ds.$$

By the above lemma,

$$\int_{-\infty}^{\infty} \delta^{(K-1)}(s)\, f'(s)\, ds = (-1)^{K-1} f^{(K)}(0).$$

$$\therefore \int_{-\infty}^{\infty} \delta^{(K)}(s)\, f(s)\, ds = (-1)^K f^{(K)}(0).$$

8/



By KVL: $y(t) + i(t) R = x(t)$

$i(t) = C\, y'(t)$.

$\Rightarrow RC\, y'(t) + y(t) = x(t)$.

a) Claim: the system is linear and shift invariant.

W.t.s. if (y_1, x_1) and (y_2, x_2) are solutions then $(y_1 + y_2,\ x_1 + x_2)$ is a solution.

$$RC\,\big(y_1'(t) + y_2'(t)\big) + \big(y_1(t) + y_2(t)\big) = x_1(t) + x_2(t)$$

$$\Leftrightarrow RC\, y_1'(t) + y_1(t) - x_1(t) = x_2(t) - RC\, y_2'(t) - y_2(t)$$

$$\Leftrightarrow 0 = 0 \quad \text{(by assumption)}.$$

W.t.s. if $(y(t), x(t))$ is a solution then $(y(t-\Delta), x(t-\Delta))$ is a solution for all $\Delta \in \mathbb{R}$.

$$RC\, y'(t-\Delta) + y(t-\Delta) = x(t-\Delta)$$

$$\Leftrightarrow RC\, y'(u) + y(u) = x(u) \quad \text{where } u = t - \Delta. \qquad \blacksquare$$

b) Find the impulse response.

First we seek the homogeneous solution. That is, we seek $y^{(h)}$ such that $(y^{(h)}, 0)$ is a solution.

$$RC\, y^{(h)\prime}(t) + y^{(h)}(t) = 0.$$

The characteristic polynomial is

$$r\,RC + 1 = 0 \Rightarrow r = {}^{-1}/_{RC}.$$

$$\Rightarrow y^{(h)}(t) = K \exp\left(\frac{-t}{RC}\right) u(t) \quad \text{where } K \in \mathbb{R}.$$

Note: we have added u into the definition of y to account for the initial conditions.

Now we seek the particular solution. We want to find h so that (h, δ) is a solution of the differential equation.

$$\delta(t) = h(t) + RC\, h'(t).$$

Integrate both sides from $-\epsilon$ to ϵ where ϵ is a small number.

$$\int_{-\epsilon}^{\epsilon} \delta(t)\, dt = \int_{-\epsilon}^{\epsilon} h(t)\, dt + RC \int_{-\epsilon}^{\epsilon} h'(t)\, dt.$$

$$\Rightarrow \quad 1 = \int_{-\epsilon}^{\epsilon} h(t)\, dt + RC \int_{-\epsilon}^{\epsilon} h'(t)\, dt.$$

We make the guess that h is integrable.

$$\Rightarrow \int_{-\epsilon}^{\epsilon} h(t)\, dt \approx 0. \quad \Rightarrow \quad 1 = RC \int_{-\epsilon}^{\epsilon} h'(t)\, dt$$

$$\Rightarrow 1 = RC\left[h(\epsilon) - h(-\epsilon)\right] = RC\, h(\epsilon)$$

since $h(-\epsilon) = 0$ by the initial conditions.

Taking the limit as $\epsilon \to 0$, $h(0) = \frac{1}{RC}$.

We can accomodate this with the homogeneous solution: $h(t) = \frac{1}{RC} \exp\left(\frac{-t}{RC}\right) u(t).$

c) $y(t) = h(t) * \Pi(t-0.5)$

d) $y(t) = h(t) * \cos(t)\, u(t)$.

9/ The differential equation that governs this system is

$$m y''(t) = b(y(t) - x(t))' + k\,(y(t) - x(t))$$

$$\Rightarrow m y''(t) - b y'(t) - k y(t) = -b x'(t) - k x(t).$$

a) First we show this system is linear and shift invariant. This is done similar to problem 8.

b) To find the impulse response, we first find the homogeneous solution.

$$m y^{(h)''}(t) - b y^{(h)'}(t) - k y^{(h)}(t) = 0.$$

Assuming the roots of the characteristic equation are unique,

$$y^{(h)}(t) = \left(B_1 e^{-\lambda_1 t} + B_2 e^{-\lambda_2 t}\right) u(t)$$

where λ_1 and λ_2 are the roots.

Now we find the particular solution.

$$m h''(t) - b h'(t) - k h(t) = -k\,\delta(t) - b\,\delta'(t).$$

Integrating both sides from $-\epsilon$ to ϵ (where ϵ is very small):

$$m \int_{-\epsilon}^{\epsilon} h''(t)\,dt - b \int_{-\epsilon}^{\epsilon} h'(t)\,dt - k \int_{-\epsilon}^{\epsilon} h(t)\,dt$$

$$= -k \int_{-\epsilon}^{\epsilon} \delta(t)\,dt - b \int_{-\epsilon}^{\epsilon} \delta'(t)\,dt.$$

From problem 7, $\int_{-\infty}^{\infty} \delta'(t)\,dt = -1 \Rightarrow \int_{-\epsilon}^{\epsilon} \delta'(t) = -1$.

$$m \int_{-\epsilon}^{\epsilon} h''(t)\,dt - b \int_{-\epsilon}^{\epsilon} h'(t)\,dt = b - k$$

where we have assumed that h is integrable and so $\int_{-\epsilon}^{\epsilon} h(t)\,dt = 0$.

$$m[h'(\epsilon)-h'(-\epsilon)] - b[h(\epsilon)-h(-\epsilon)] = b-k$$

$$\Rightarrow m\,h'(\epsilon) - b\,h(\epsilon) = b-k. \qquad \cdots\cdots \text{①}$$

We have an equation with two unknowns: $h(\epsilon)$, $h'(\epsilon)$.

Consider again $m\,h''(t) - b\,h'(t) - k\,h(t) = -k\delta(t) - b\delta'(t)$.

Integrate both sides from $-\infty$ to t.

$$m\int_{-\infty}^{t} h''(\tau)\,d\tau - b\int_{-\infty}^{t} h'(\tau)\,d\tau - k\int_{-\infty}^{t} h(\tau)\,d\tau$$
$$= -k\int_{-\infty}^{t} \delta(\tau)\,d\tau - b\int_{-\infty}^{t}\delta'(\tau)\,d\tau.$$

If $t<0$ then we get $0=0$.

If $t>0$, $m\,h'(t) - b\,h(t) - k\,g(t) = -k\,u(t) - b\,\delta(t)$

where $g(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$.

Now integrate both sides from $-\epsilon$ to ϵ:

$$m\int_{-\epsilon}^{\epsilon} h'(t)\,dt - b\int_{-\epsilon}^{\epsilon} h(t)\,dt - k\int_{-\epsilon}^{\epsilon} g(t)\,dt = -k\int_{-\epsilon}^{\epsilon} u(t)\,dt$$
$$- b\int_{-\epsilon}^{\epsilon} \delta(t)\,dt$$

$\Rightarrow m\,h(\epsilon) = -k\epsilon - b \approx -b$ since ϵ is very small.

$\Rightarrow h(\epsilon) = -b/m$.

From ①, $m\,h'(\epsilon) - b\,h(\epsilon) = b-k \Rightarrow h'(\epsilon) = \frac{b}{m} - \frac{k}{m} - \frac{b^2}{m^2}$.

Can we satisfy these values of $h(\epsilon)$ and $h'(\epsilon)$ with the form of the homogeneous solution? Let's suppose so and see what happens.

$$h(t) = \left(B_1 e^{-\lambda_1 t} + B_2 e^{-\lambda_2 t}\right)u(t).$$

$$\left.\begin{aligned} h(\epsilon) &= B_1 e^{-\lambda_1\epsilon} + B_2 e^{-\lambda_2\epsilon} \approx B_1 + B_2. \\ h'(\epsilon) &= -B_1\lambda_1 e^{-\lambda_1\epsilon} - B_2\lambda_2 e^{-\lambda_2\epsilon} \approx -B_1\lambda_1 - B_2\lambda_2. \end{aligned}\right\}$$

Here we have two equations with two unknowns. Solving for B_1 and B_2 determines the form of h!