EE 102A - Assignment 4

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Justify all your answers.

Problem 1. The Fourier Transform accepts a function as input and outputs another function. It is a system! Determine if the Fourier Transform is linear, shift invariant, causal, has memory.

Problem 2. Find the Fourier transform of $f(s) = \exp(-a\,|s|)$ where $a > 0$.

Problem 3. Consider a causal linear (but not necessarily shift invariant) system S. What must be true of its impulse responses?

Problem 4. The average value of a function $f : [0, T] \to \mathbb{C}$ is defined as

$$\text{mean}(f) = \frac{1}{T} \int_0^T f(\gamma)\, d\gamma.$$

How can we calculate the average value of a function f from its Fourier Series coefficients without synthesizing the function and integrating?

Problem 5. You are provided with the step response $\nu = S\{u\}$ (where u is the Heaviside step function) of a linear shift invariant system S. How can you determine the impulse response of the system from its step response?

Problem 6. *Fourier Symmetry*
Prove the following properties:
a) The Fourier Transform of a function that is real and even is real and even.
b) The Fourier Transform of a function that is real and odd is imaginary and odd.
c) The Fourier Transform of a real function is Hermitian symmetric.
d) The magnitude of the Fourier Transform of a real function is even.
e) The phase of the Fourier Transform of a real function is odd.

Problem 7. Prove the Fourier Shift theorem, which states

$$\mathcal{F}\{f(x-a)\}(k) = \exp(-i\,2\pi\,k\,a)\,\mathcal{F}\{f\}(k).$$

Problem 8. Let $*$ represent convolution.
a) Find a simple expression for $\text{sinc} * \text{sinc}$ and prove that you are correct.
b) Find a simple expression for $\underbrace{\text{sinc} * \text{sinc} * \cdots * \text{sinc}}_{N \text{ times}}$ and prove that you are correct.

The function sinc is defined as

$$\text{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

for all x.

Problem 9.
a) The current-voltage relationship of an inductor is

$$v(t) = L\frac{di}{dt}(t),$$

where v is the voltage applied, i is the current going through the inductor, and $L \in \mathbb{R}$ is called the inductance. If the current is considered the input the voltage applied is considered the output, derive the transfer function of this system.
b) The current-voltage relationship of a capacitor is

$$i(t) = C\frac{dv}{dt}(t),$$

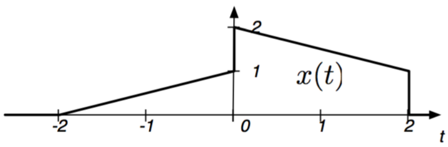
where $C \in \mathbb{R}$ is the capacitance. If the current is considered the input and the voltage applied is considered the output, derive the transfer function of this system.

Problem 10. The cross-correlation of f and g is defined as follows:

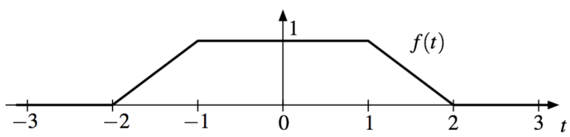
$$(f \star g)(x) = \int_{-\infty}^{\infty} \overline{f}(\gamma)g(\gamma + x)d\gamma.$$

Find the Fourier Transform of $f \star g$ in terms of the Fourier transforms of f and g.

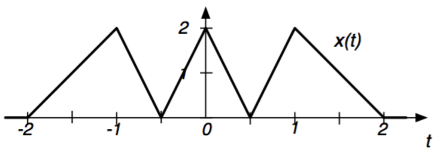
Problem 11. *Fourier Series Approximation*
Approximate each of the following functions with 1, 3, 5, 9, 17 terms of the Fourier Series and plot the result. Consider using the `subplot` command to make your plotting nicer.
a) Square wave with a period of 1 and an amplitude of 1.
b) Sawtooth wave with a period of 1 and an amplitude of 1.
c) Triangle wave with a period of 1 and an amplitude of 1.
d) The function $x : [-2, 2] \to \mathbb{R}$ shown below:

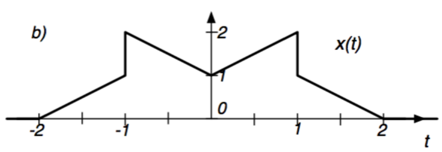


Problem 12. *Exploiting Linearity*
a) Express the following function as a linear combination of simpler functions.

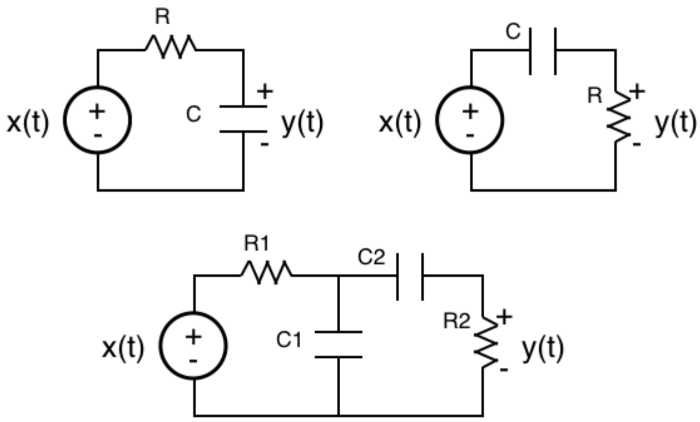


b) Determine the Fourier Transform of this function.
c) Do the same for the following two functions.



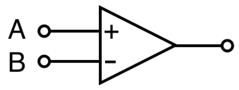


Problem 13. *Circuit Systems*
Find the transfer functions of the three circuits shown below (where x is the input and y is the output).

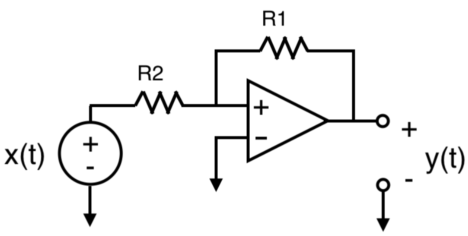


Plot the modulation transfer functions. What does each circuit do (i.e. how does each circuit affect the spectrum of the incoming signal)? Any idea what these circuits are called?

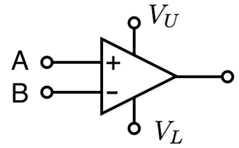
Problem 14. *Ideal Operational Amplifier*
The figure below shows the schematic depiction of an operational amplifier (op-amp). For an ideal op-amp, the currents going through the wires of terminals A and B are always 0, and the voltage at terminal A is the same as the voltage at terminal B.



a) Consider the circuit shown below where x is the input and y is the output. Is this a linear shift-invariant system? Find the transfer function for the circuit shown below (where x is the input and y is the output). What does this circuit do?



b) One aspect of a real op-amp is that the output voltage can never exceed the "rail" voltages. The rails are sometimes depicted as shown below. In this figure, the output voltage can never exceed the upper rail V_U, and can never go below the lower rail V_L.



Suppose the upper and lower rails are specified to be $\pm V_{CC}$. Is the circuit above a linear shift-invariant system? If not, can you restrict the domain of input functions so that the system is a linear shift invariant system?

Plot the output when $x(t) = 2\cos(2\pi t)\,V$, $R_1 = 20\,\Omega$, $R_2 = 10\,\Omega$, and $V_{CC} = 3.5\,V$.

Problem 15. *System Properties*

Determine whether each of the following systems is linear or non-linear, shift invariant or shift variant, causal or non-causal, and has memory or is memoryless.

a) $y(\gamma) = x(\gamma)\,\exp(i2\pi f\gamma + \phi)$
b) $y(t) = \frac{d}{dt}\,(t\,x(t))$
c) $x(z) = 1 + y(z)$
d) $z(y) = 1 - \cos^2(i\omega y + x(y))$
e) $\gamma(t) = x(t)\,\cos^2(t/2)$
f) $y(f) = \int_{-\infty}^{f+1} x(\nu)\,d\nu$
g) $k(a) = x\,(\sin(a))$