Assignment 4 - Solutions

1) Claim: the Fourier Transform is linear.

Proof:

1) Superposition:

$$\tilde{\mathfrak{F}}\{f+g\}(k) = \int_{-\infty}^{\infty} (f+g)(x)\, e^{-i2\pi kx}\, dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi kx}\, dx + \int_{-\infty}^{\infty} g(x)\, e^{-i2\pi kx}\, dx$$

$$= \tilde{\mathfrak{F}}\{f\}(k) + \tilde{\mathfrak{F}}\{g\}(k).$$

2) Homogeneity:

$$\tilde{\mathfrak{F}}\{\alpha f\}(k) = \int_{-\infty}^{\infty} (\alpha f)(x)\, e^{-i2\pi kx}\, dx$$

$$= \alpha \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi kx}\, dx = \alpha\, \tilde{\mathfrak{F}}\{f\}(k).$$

$\tilde{\mathfrak{F}}\{\delta\} = 1 \Rightarrow$ the Fourier Transform is non-causal and has memory.

Recall the shift theorem:

$$\tilde{\mathfrak{F}}\{f(x-d)\}(k) = e^{-i2\pi kd}\, \tilde{\mathfrak{F}}\{f\}(k) \neq \tilde{\mathfrak{F}}\{f\}(k-\Delta).$$

$\therefore\ \tilde{\mathfrak{F}}$ is shift variant.

2) $\tilde{\mathfrak{F}}\{f(s)\}(k) = \tilde{\mathfrak{F}}\{f_L(s)\}(k) + \tilde{\mathfrak{F}}\{f_R(s)\}(k)$

where $f_R(s) = e^{-as}u(s)$ and $f_L(s) = f_R(-s) = f_R^-(s)$.

$$\tilde{\mathfrak{F}}\{f_R(s)\}(k) = \int_{-\infty}^{\infty} f_R(s)\, e^{-i2\pi ks}\, ds = \int_{-\infty}^{\infty} e^{-as}u(s)\, e^{-i2\pi ks}\, ds$$

$$= \int_{0}^{\infty} e^{-(a+i2\pi k)s}\, ds = \frac{-1}{a+i2\pi k}\, e^{-(a+i2\pi k)s}\Big|_0^{\infty} = \frac{1}{a+i2\pi k}.$$

$$\tilde{\mathfrak{F}}\{f_L\}(k) = \tilde{\mathfrak{F}}\{f_R^-\}(k) = \overline{\tilde{\mathfrak{F}}\{f_R\}}(k) = \frac{1}{a-i2\pi k}.$$

$$\Rightarrow \tilde{\mathfrak{F}}\{f\}(k) = \frac{1}{a+i2\pi k} + \frac{1}{a-i2\pi k}.$$

3) For a linear system S, the output for some input x can be expressed using the Superposition Integral:

$$S\{x\}(\gamma) = \int_{-\infty}^{\infty} x(s)\, h_\gamma(s)\, ds.$$

If S is causal, then $h_\gamma(s) = 0 \ \forall\, s < 0$ and all $\gamma \in \mathbb{R}$.

4) The Fourier Series coefficients are

$$F_n = \frac{1}{T} \int_0^T f(x) \exp\left(i 2\pi \frac{nx}{T}\right) dx.$$

Consider $n = 0$:

$$F_0 = \frac{1}{T} \int_0^T f(x)\, dx.$$

F_0 is the mean value of f.

5) Consider a system D such that $D\{x\} = x'$. Both D and S are LSI systems.

We know $D\{u\} = \delta$, where u is the step function.

$$\Rightarrow \quad u \to \boxed{D} \xrightarrow{\ \delta\ } \boxed{S} \to h$$

where h is the impulse response of S.

Since D and S are LSI, they commute.

$$\Rightarrow \quad u \to \boxed{S} \longrightarrow \boxed{D} \to h.$$

This shows that taking the derivative of the step response yields the impulse response.

6/ a) Claim: $\tilde{\mathcal{F}}\{\text{real and even}\}$ = real and even.
Proof:
Let f be real and even.

$$\tilde{\mathcal{F}}\{f\}(k) = \int_{-\infty}^{\infty} f(x)\left[\cos(2\pi kx) - i\sin(2\pi kx)\right]dx$$
$$= \int_{-\infty}^{\infty} f(x)\cos(2\pi kx)\,dx - i\int_{-\infty}^{\infty} f(x)\sin(2\pi kx)\,dx.$$

Since f is even and sine is odd, f·sine is odd.

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\sin(2\pi kx)\,dx = 0.$$

$$\Rightarrow \tilde{\mathcal{F}}\{f\}(k) = \int_{-\infty}^{\infty} f(x)\cos(2\pi kx)\,dx.$$

f·cos is even. An integration of even functions is even. ∎

b) Similar to (a).

c) Combining (a) and (b) proves this.

d) Claim: The magnitude of the Fourier Transform of a real function is even.
Proof:
W.t.s. $|\tilde{\mathcal{F}}\{f\}(k)| = |\tilde{\mathcal{F}}\{f\}(-k)|$.

From (c), we know $\tilde{\mathcal{F}}\{f\}(-k) = \overline{\tilde{\mathcal{F}}\{f\}(k)}$.

$\therefore\ |\tilde{\mathcal{F}}\{f\}(-k)| = |\tilde{\mathcal{F}}\{f\}(k)|$.

e) Similar to (d).

8/ a) $\tilde{\mathcal{F}}\{\text{sinc} * \text{sinc}\} = \Pi \cdot \Pi = \Pi$.

$\Rightarrow \tilde{\mathcal{F}}^{-1}\{\tilde{\mathcal{F}}\{\text{sinc} * \text{sinc}\}\} = \text{sinc} * \text{sinc} = \tilde{\mathcal{F}}^{-1}\{\Pi\} = \text{sinc}$.

b) Prove by induction using (a).

9/ a) $v(t) = L\, i'(t)$.

Taking the Fourier Transform of both sides:

$$V(k) = L\,(i\,2\pi k)\,I(k).$$

$$\Rightarrow H(k) = \frac{V(k)}{I(k)} = i L 2\pi k.$$

b) $i(t) = C\, v'(t) \Rightarrow I(k) = C\,(i 2\pi k)\, V(k)$.

$$\Rightarrow H(k) = \frac{V(k)}{I(k)} = \frac{1}{i C 2\pi k}.$$

10/ $(f \star g)(s) = \bar{f}(-s) * g(s)$.

$$\Rightarrow \mathcal{F}\{f \star g\} = \bar{F}\, G.$$

12/ a) $f(t) = 2\Lambda(t/2) - \Lambda(t)$.

$$\Rightarrow \mathcal{F}\{f\}(s) = 4\,\mathrm{sinc}^2(2s) - \mathrm{sinc}^2(s).$$

b) $x(t) = 4\Lambda(t/2) - 6\Lambda(t) + 4\Lambda(2t)$.

$$\Rightarrow \mathcal{F}\{x\}(\gamma) = 8\,\mathrm{sinc}^2(2\gamma) - 6\,\mathrm{sinc}^2(\gamma) + 2\,\mathrm{sinc}^2(\gamma/2).$$

c) $x(t) = 2\Lambda(t/2) - 2\Lambda(t) + \Pi(t/2)$.

$$\Rightarrow \mathcal{F}\{x\}(k) = 4\,\mathrm{sinc}^2(2\gamma) - 2\,\mathrm{sinc}^2(\gamma) + 2\,\mathrm{sinc}(2\gamma).$$

13/ a) $H(k) = \dfrac{1}{1 + jRC2\pi k}$.

b) $H(k) = \dfrac{RC(j2\pi k)}{1 + RC(j2\pi k)}$

c) $$H(f) = \frac{Z_{c1}(f)\,R_2}{Z_{c1}(f)\,Z_{c2}(f) + Z_{c2}(f)R_1 + Z_{c1}(f)R_2 + Z_{c1}(f)R_1 + R_1R_2}$$

where $Z_{c1}(f) = \dfrac{1}{j2\pi f C_1}$, $Z_{c2}(f) = \dfrac{1}{j2\pi f C_2}$.

14| a) $y(t) = \frac{-R_1}{R_2} x(t)$. $H(k) = {}^{-R_1}/_{R_2}$.

This is an LSI system.

15| a) Linear, Causal, Memoryless, and shift variant.

b) Linear, Causal, ~~Memoryless~~ Has memory, and shift variant.

c) Non-linear, causal, memoryless, shift-invariant.

d) Non-linear, causal, memoryless, shift-variant.

e) Linear, causal, memoryless, shift-variant.

Proof of memoryless:

$$\cos^2(t/2) = 1/2\,(1+\cos(t)).$$

$$\Rightarrow y(t) = x(t)\,(1/2)\,(1+\cos(t)).$$

f) Linear, Non-causal, has memory, shift-invariant.

Proof of shift invariant:

$$S\{x(t-\Delta)\}(f) = \int_{-\infty}^{f+1} x(\nu-\Delta)\,d\nu.$$

Let $u = \nu - \Delta$. $\Rightarrow du = d\nu$.

$$\Rightarrow S\{x(t-\Delta)\}(f) = \int_{-\infty}^{f+1-\Delta} x(u)\,du.$$

g) Linear, Non-causal, has memory, Shift variant.