Assignment 7 - Solutions.

1) $\mathrm{DFT}\{x\}[m] = \sum_{n=0}^{N-1} x[n] \exp\left(-i 2\pi \frac{mn}{N}\right)$.

DFT is a linear transformation from a finite dimensional vector space to a finite dimensional vector space. And thus, it can be represented as matrix multiplication.
That is, there exists a matrix F such that
$y = Fx \Leftrightarrow y = \mathrm{DFT}\{x\}$.

Each column of this matrix can be found by computing the DFT of the standard basis vectors:

$$F = \begin{bmatrix} | & | & & | \\ \mathrm{DFT}\{e_1\} & \mathrm{DFT}\{e_2\} & \cdots & \mathrm{DFT}\{e_N\} \\ | & | & & | \end{bmatrix}$$

where $e_i \in \mathbb{R}^N$ such that the i^{th} element of e_i is 1 and all other elements are 0.

So the first column of F is

$$\mathrm{DFT}\left\{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$ Similarly for the rest of the columns.

Thus,

$$F = \begin{bmatrix} 1 & W^0 & \cdots & W^0 \\ 1 & W^1 & & W^{N-1} \\ \vdots & \vdots & \ddots & W^{2(N-1)} \\ \vdots & \vdots & & \vdots \\ 1 & W^{N-1} & \cdots & W^{(N-1)^2} \end{bmatrix}$$ where $W = \exp(-i 2\pi/N)$.

Similarly for the IDFT.

It's easy to show, then, that $F^{-1} = \frac{1}{N} F^*$ by seeing the matrix for the IDFT and its relationship to F.

2) $H(f) = \dfrac{Y(f)}{X(f)}$ where $x \to \boxed{S} \to y$.

Suppose $x = \Pi \Rightarrow X = \text{sinc}$. Then X has zeros at integer values in its domain.

We can find almost all values of H, though, by computing

$$H(f) = \frac{S\{\Pi\}(f)}{\text{sinc}(f)}$$

for those values of f where $\text{sinc}(f) \neq 0$.

Then we can conduct another experiment where ~~$x(t) = \text{sinc}(\pi t)$. $\Rightarrow X(f) = \pi$~~

$$x(t) = \Pi(t/\pi) \Rightarrow X(f) = \pi \, \text{sinc}(\pi f).$$

Then we can compute

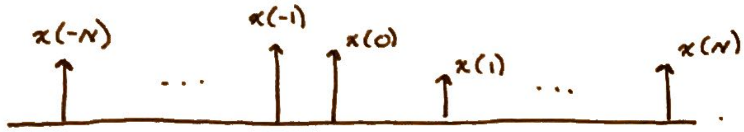
$$H(f) = \frac{S\{\Pi(t/\pi)\}(f)}{\pi \, \text{sinc}(\pi f)}$$

for all those values of f where $\text{sinc}(f) = 0$.

This gives us all values of H.

(Note: in reality, there would be noise. But that is outside the scope of this class.)

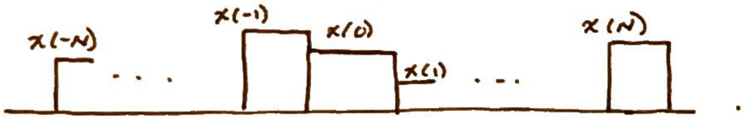
3) We can represent our samples at points as



That is we can create a weighted delta train:

$$x_s(t) = \sum_{n=-N}^{N} x(n)\,\delta(t-n).$$

a) The zero-order-hold function is



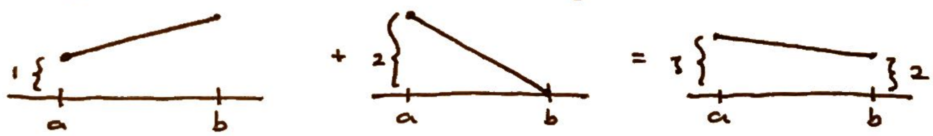
This can be created as $x_{ZOH}(t) = x_s(t) * \Pi(t-\tfrac{1}{2})$.

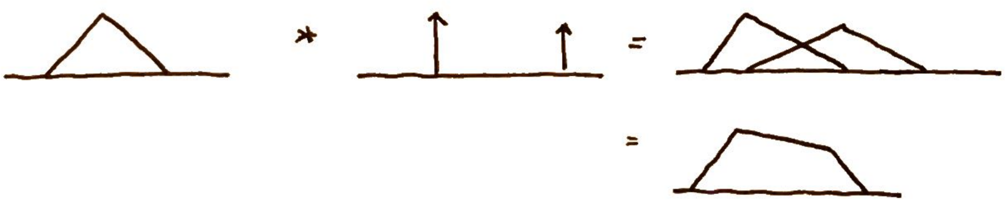
$$\Rightarrow x_{ZOH}(t) = \sum_{n=-N}^{N} x(n)\,\Pi(t-n-\tfrac{1}{2}),$$

which is exactly what we wanted.

Since zero-order-hold is convolution with a shifted rect, this corresponts to multiplication of the spectrum with a sinc and a linear phase.

b) Recall that the addition of two line segments is a line segment.





This is linear interpolation.

$$x_L(t) = x_s(t) * \Lambda(t).$$

This corresponds to multiplication of the spectrum by a sinc^2.

4) a) $[0 \;\; 1/8 \;\; 2/8 \cdots 7/8]$. b) $[-4/8 \cdots -1/8 \;\; 0 \;\; 1/8 \cdots 3/8]$.

c) $[0 \;\; 1/7 \cdots 6/7]$. $[-3/7 \;\; -2/7 \;\; -1/7 \;\; 0 \;\; 1/7 \cdots 3/7]$.

5) $$\tilde{\mathcal{F}}\{Ш_\Delta(x)\}(k) = \tilde{\mathcal{F}}\left\{\sum_{n=-\infty}^{\infty} \delta(x-n\Delta)\right\}(k) = \tilde{\mathcal{F}}\left\{\sum_{n=-\infty}^{\infty} \delta\left(\Delta\left(\frac{x}{\Delta}-n\right)\right)\right\}(k)$$

$$= \frac{1}{\Delta}\tilde{\mathcal{F}}\left\{\sum_{n=-\infty}^{\infty}\delta\left(\frac{x}{\Delta}-n\right)\right\}(k) = \frac{1}{\Delta}\tilde{\mathcal{F}}\{Ш(x/\Delta)\}(k)$$

$$= Ш(k\Delta) = \sum_{n=-\infty}^{\infty}\delta(k\Delta-n) = \frac{1}{\Delta}\delta\left(k-\frac{n}{\Delta}\right) = \frac{1}{\Delta}Ш_{1/\Delta}(k).$$

6) $f = f_R + i f_I$ where $f_R = Re\{f\}$ and $f_I = Im\{f\}$.

$f_R = f_R^{(e)} + f_R^{(o)}$ and $f_I = f_I^{(e)} + f_I^{(o)}$

where (e) and (o) denote the even and odd components respectively.

$$F = \tilde{\mathcal{F}}\{f\} = \tilde{\mathcal{F}}\{f_R^{(e)} + f_R^{(o)} + i f_I^{(e)} + i f_I^{(o)}\}$$

$$= \tilde{\mathcal{F}}\{f_R^{(e)}\} + \tilde{\mathcal{F}}\{f_R^{(o)}\} + i\tilde{\mathcal{F}}\{f_I^{(e)}\} + i\tilde{\mathcal{F}}\{f_I^{(o)}\}.$$

Note that $f_R^{(e)}$, $f_R^{(o)}$, $f_I^{(e)}$, and $f_I^{(o)}$ are all real.

$\tilde{\mathcal{F}}\{f_R^{(e)}\}$ is real and even. $\tilde{\mathcal{F}}\{f_R^{(o)}\}$ is imaginary and odd.
$\tilde{\mathcal{F}}\{f_I^{(e)}\}$ is real and even $\Rightarrow i\tilde{\mathcal{F}}\{f_I^{(e)}\}$ is imaginary and even.
$\tilde{\mathcal{F}}\{f_I^{(o)}\}$ is imaginary and odd $\Rightarrow i\tilde{\mathcal{F}}\{f_I^{(o)}\}$ is real and odd.

$\therefore Re(F) = \tilde{\mathcal{F}}\{f_R^{(e)}\} + i\tilde{\mathcal{F}}\{f_I^{(o)}\}$.

Let $g = \tilde{\mathcal{F}}^{-1}\{Re\{\tilde{\mathcal{F}}\{f\}\}\} = \tilde{\mathcal{F}}^{-1}\{\tilde{\mathcal{F}}\{f_R^{(e)}\} + i\tilde{\mathcal{F}}\{f_I^{(o)}\}\}$

$\cancel{= f_R^{(e)} + i f} = f_R^{(e)} + i f_I^{(o)}$. $\Rightarrow g^{(e)} = f_R^{(e)}$.

$$g^{(e)}(x) = f_R^{(e)}(x) = \frac{f_R(x) + f_R(-x)}{2}. \Rightarrow g^{(e)}(x) = \tfrac{1}{2}f_R(x) \;\; \forall \; x \geq 0.$$

$$g^{(o)}(x) = i f_I^{(o)}(x) = i\left[\frac{f_I(x) - f_I(-x)}{2}\right] \Rightarrow g^{(o)}(x) = \tfrac{i}{2}f_I(x) \;\; \forall \; x \geq 0.$$

7/ Let $g(t) = \Pi(t/\epsilon) * f(t)$.

Then our samples are samples of g.
So with sinc interpolation, we reconstruct g.

$$\mathcal{F}\{g\}(k) = \epsilon \operatorname{sinc}(\epsilon k) \cdot \mathcal{F}\{f\}(k).$$