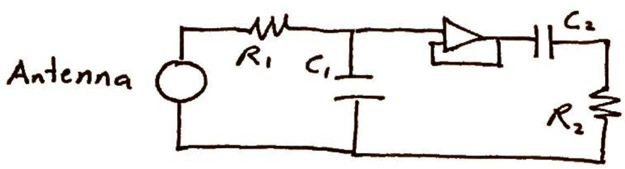
Assignment 9 - Solutions

1/ Student A is incorrect. In his/her statement, f has compact support. But in the previous homework, f had compact domain.

2/ Since the sampling frequency is 10khz, the pre-filter should be a low-pass filter with bandwidth less than 5 khz.



3/ a) $E_f = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\overline{f(x,y)}\,dx\,dy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\overline{\tilde{\mathcal{F}}_{2D}^{-1}\{F(k_x,k_y)\}(x,y)}\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \iint_{-\infty}^{\infty} \overline{F(k_x,k_y)}\, e^{-i2\pi(k_x x + k_y y)}\,dk_x\,dk_y\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{F(k_x,k_y)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi(k_x x + k_y y)}\,dx\,dy\,dk_x\,dk_y$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{F(k_x,k_y)}\, F(k_x,k_y)\,dk_x\,dk_y = E_F$$

b) $\tilde{\mathcal{F}}\{f(x-a,y-b)\}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-a,y-b)\, e^{-i2\pi(xu+yv)}\,dx\,dy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tilde{u},\tilde{v}) \exp\left(-i2\pi\left(u(\tilde{u}+a) + v(\tilde{v}+b)\right)\right) d\tilde{u}\,d\tilde{v}$$

$$= e^{-i2\pi(ua+vb)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tilde{u},\tilde{v})\, e^{-i2\pi(u\tilde{u}+v\tilde{v})}\,d\tilde{u}\,d\tilde{v}$$

$$= e^{-i2\pi(ua+vb)}\, \tilde{\mathcal{F}}\{f\}(u,v).$$

c) $\tilde{\mathcal{F}}\{f(ax,by)\}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(ax,by)\, e^{-i2\pi(ux+vy)}\,dx\,dy$

$$= \frac{1}{ab}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tilde{x},\tilde{y}) \exp\left(-i2\pi\left(u\frac{\tilde{x}}{a} + v\frac{\tilde{y}}{b}\right)\right) d\tilde{x}\,d\tilde{y}$$

$$= \frac{1}{ab}\,\tilde{\mathcal{F}}\{f\}\left(\frac{u}{a},\frac{v}{b}\right).$$

4] The key is to realize that f is separable.

$$f(x,y) = M \Lambda\left(\frac{x - G/2}{G/2}\right) \Pi\left(\frac{y - K/2}{K}\right).$$

From here you can use the scale and shift theorems:

$$\tilde{\mathcal{F}}\{f\}(u,v) = M\, \tilde{\mathcal{F}}_{1D}\left\{\Lambda\left(\frac{x - G/2}{G/2}\right)\right\}(u)\; \tilde{\mathcal{F}}_{1D}\left\{\Pi\left(\frac{y-K/2}{K}\right)\right\}(v).$$

7] $S\{x(t)\}(\gamma) = \int_{-\infty}^{\gamma} x(\tau)\,d\tau + 8$

This is a non-linear system. However, we can recognize that $\tilde{S}\{x\}(\gamma) = \int_{-\infty}^{\gamma} x(\tau)\,d\tau = (x * u)(\gamma)$

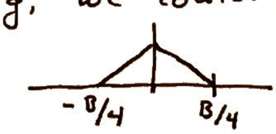
is indeed a linear system. Note that u is the Heaviside step function.

Let $\tilde{y} = \tilde{S}\{x\}(\gamma)$. $\Rightarrow$ $\tilde{\mathcal{F}}\{\tilde{y}\} = \tilde{\mathcal{F}}\{x\}\,\tilde{\mathcal{F}}\{u\}$.

$\Rightarrow$ $\tilde{\mathcal{F}}\{y\} = \tilde{\mathcal{F}}\{x\}\,\tilde{\mathcal{F}}\{u\} + 8$.

This shows us that it can often be useful to incorporate linear system theory into problems involving non-linear systems.

8] By multiplying with the appropriate cosine and then low-pass filtering, we could get the following spectrum:



Now the sampling rate only needs to be greater than $B/2$ for perfect reconstruction with sinc interpolation (according to the Nyquist theorem).

9) Applying a linear phase in the frequency domain can be thought of as a shift in the time domain. In this way, we can implement shifts of non-integer value.

10)