Signal Processing and Linear Systems1

Lecture 2: The Basics

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Outline

Complex Numbers

Vectors

Functions

Real numbers have a set of properties:

- There exists 0 such that x + 0 = x for all x
- There exists 1 such that 1 x = 1 for all x
- For every x there exists -x such that x + (-x) = 0
- For every nonzero x there exists 1/x such that (x)(1/x) = 1
- (a + b) + c = a + (b + c)
- (a b) c = a (b c)
- a + b = b + a
- a b = b a
- a (b + c) = a b + a c
- For any a and b, either a < b, a = b, or a > c

These properties give us a lot of power

Suppose, instead, we had pairs of numbers? e.g. $(2, 8)$ (π, e)

We need to define addition and multiplication for these pairs of numbers.

How close can we come with pairs of numbers?
How many of those properties can we retain?

What if we define addition and multiplication pointwise?
(a,b) + (c,d) = (a + c, b + d) (a,b) (c,d) = (a c, b d)

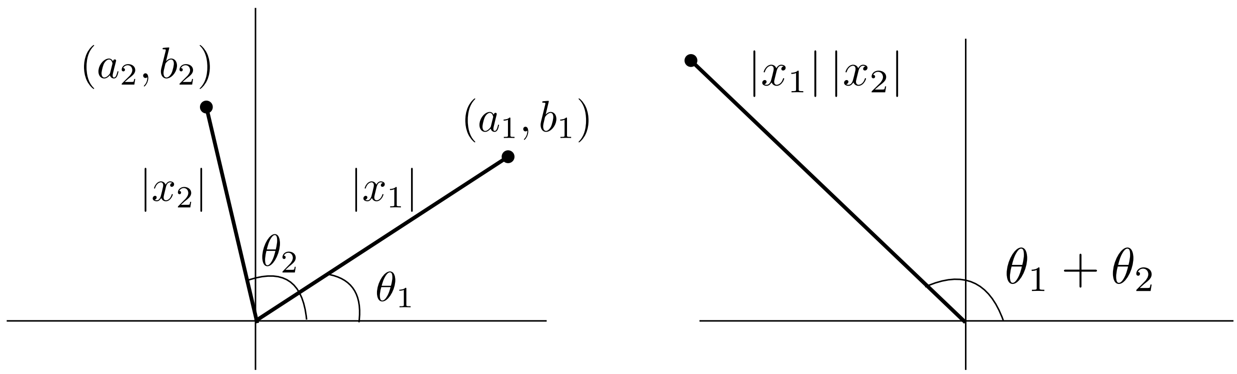
We lose one of the properties that we like very much.

(1,0) is not equal to 0, but (1,0) doesn't have a multiplicative inverse

"For every nonzero x there exists 1/x such that (x)(1/x) = 1" doesn't apply.

Complex Multiplication

Plot two points $x_1 = (a_1, b_1)$ and $x_2 = (a_2, b_2)$ as vectors on a graph.



Addition is defined pointwise.

With these definitions of addition and multiplication, we get back as many properties as we can.

Which properties did we lose?

Terminology

$$x = (a, b) = a + b\,i = |x| \angle\, \theta$$

a is called the real part of x

b is called the imaginary part of x

$|x|$ is called the magnitude of x

θ is called the phase of x

Euler's Formula

$$\exp{(i\theta)} = \cos\theta + i\sin\theta$$

Euler's Identity

$$e^{i\,\pi} = -1$$

Conjugate

$$x = (a, b) = a + i\,b$$

$$\bar{x} = (a, b) = a - i\,b$$

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Vectors

A vector is an ordered finite list of numbers together with pointwise addition and scalar multiplication.

Example: $\begin{bmatrix} -1.1 \\ 0.0 \\ 3.6 \\ -7.2 \end{bmatrix}$ $(-1.1, 0.0, 3.6, -7.2)$

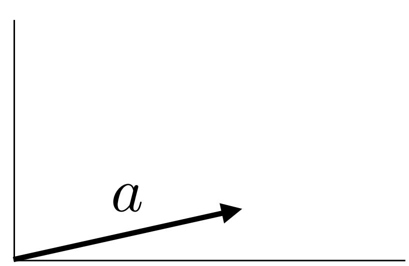
Example: 0

All the elements are 0.

The length is understood from context.

Drawing Vectors in 2D

$$a = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$



Vector Addition

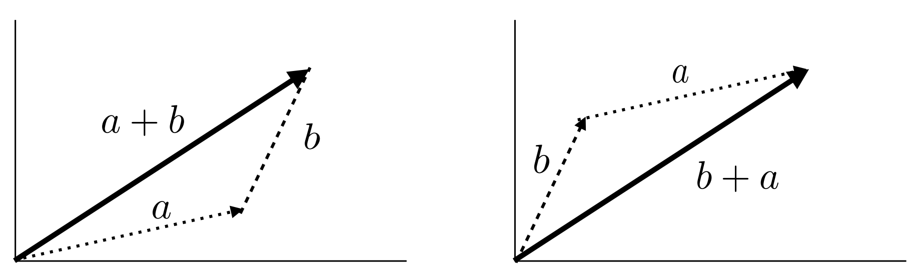
Two vectors of the same size can be added together by adding corresponding components.

Example: $$\begin{bmatrix}0\\7\\3\end{bmatrix} + \begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}1\\9\\3\end{bmatrix}$$

Example: $$\begin{bmatrix}1\\9\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\8\end{bmatrix}$$

Geometric Interpretation

Vectors add tip-to-tail.



Scalar Multiplication

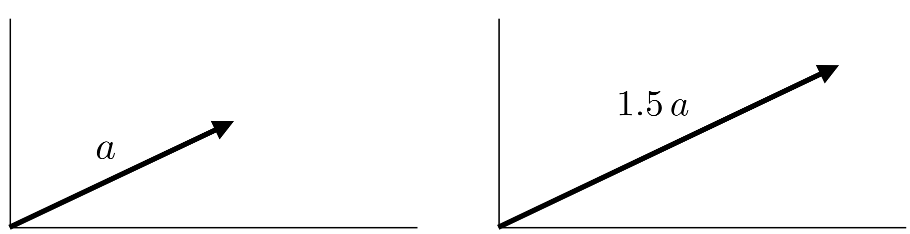
Every element of the vector is multiplied by the scalar (i.e. number)

Example:

$$(-2)\begin{bmatrix} 1 \\ 9 \\ -6 \end{bmatrix} = \begin{bmatrix} -2 \\ -18 \\ 12 \end{bmatrix}$$

Geometric Interpretation

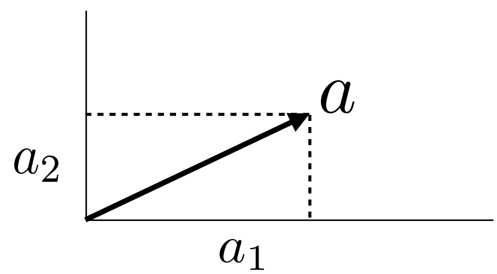
Vector is scaled by scalar multiplication.



L2 Norm

The L2 norm of a vector a , denoted by $\|a\|_2$, is

$$\|a\|_2 = \sqrt{|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2}$$



Another name for the L2 norm is “length.”

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Dot Product

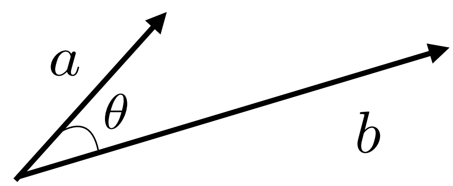
If a and b are vectors of complex numbers then

$$a \cdot b = b^* \, a = a_1 \, \bar{b}_1 + a_2 \, \bar{b}_2 + \cdots + a_n \, \bar{b}_n$$

Angle Between Two Vectors

Let θ denote the angle between vectors a and b .

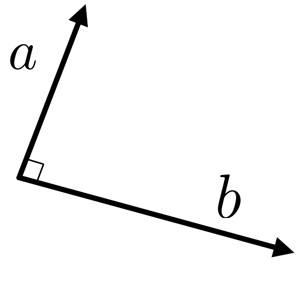
$$\theta = \arccos\left(\frac{a \cdot b}{\|a\|_2 \, \|b\|_2}\right)$$



Perpendicular Vectors

Two vectors a and b are perpendicular if and only if

$$a \cdot b = 0$$



Dot Product Properties

The angle between two vectors *a,b* is acute if and only if

$$a \cdot b > 0$$

The angle between two vectors *a,b* is obtuse if and only if

$$a \cdot b < 0$$

Linear Combination

Suppose $a_1, a_2, \ldots, a_n$ are vectors of the same size.

A linear combination of these vectors is an expression of the form

$$\beta_1 \, a_1 + \beta_2 \, a_2 + \cdots + \beta_n \, a_n$$

where $\beta_1, \beta_2, \ldots, \beta_n$ are scalars.

Linearly Independent

A set of vectors $a_1, a_2, \ldots, a_n$ is Linearly Independent means the only solution to

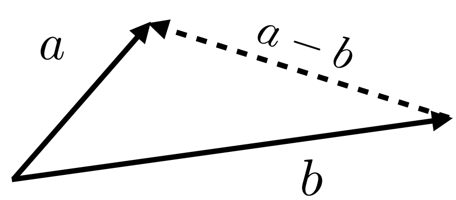
$$c_1\, a_1 + c_2\, a_2 + \cdots + c_n\, a_n = 0$$

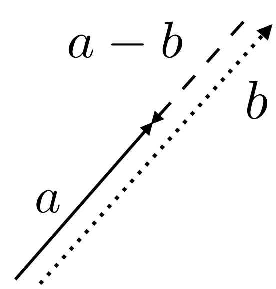
is $c_1 = c_2 = \cdots = c_n = 0$

Metric of Similarity - L2 Norm

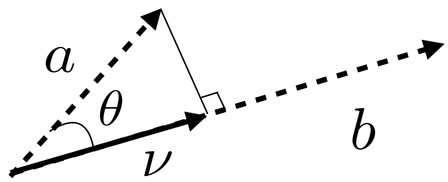
If the L2 norm = 0, the vectors are identical

$$\|a - b\|_2$$





Vector Projection



ν is called the projection of vector a onto b .

$$\nu = \text{proj}_b a = \frac{a \cdot b}{||b||_2^2} b$$

Outline

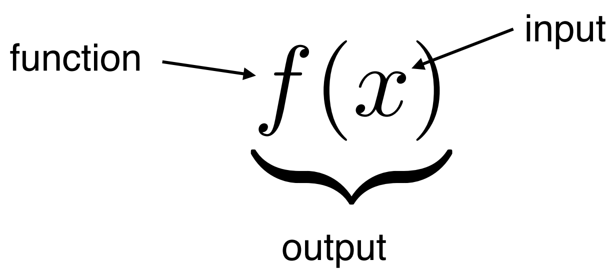
Complex Numbers

Vectors

Functions

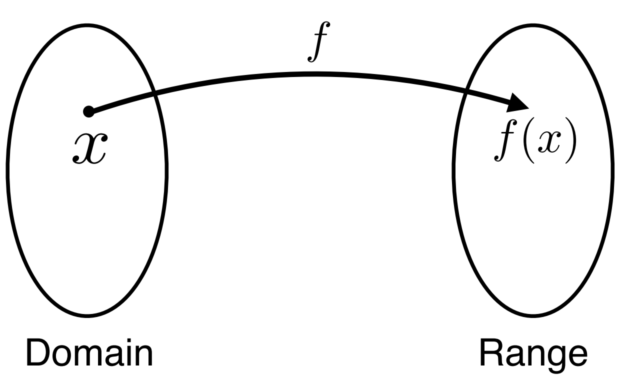
Functions

A function is a mathematical machine
  You input something
  You get something out



As long as you input the same thing, you'll always get the same thing out.

Consider a function f. What is the precise definition of a function?

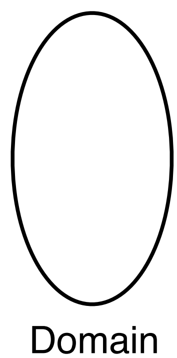


As the picture above indicates, a function has three parts. We define a function as an ordered triple.

Domain

A function is an ordered triple.

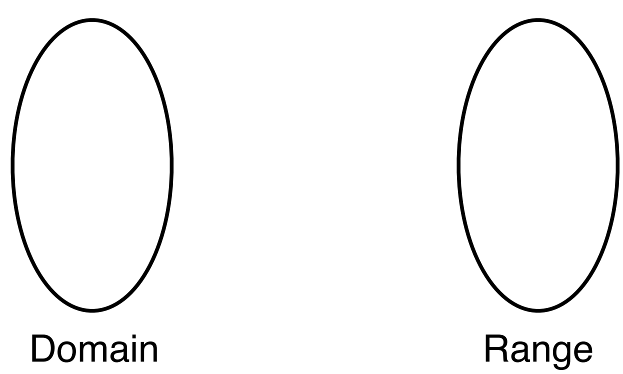
The first element is a set called the Domain.



Range

A function is an ordered triple.

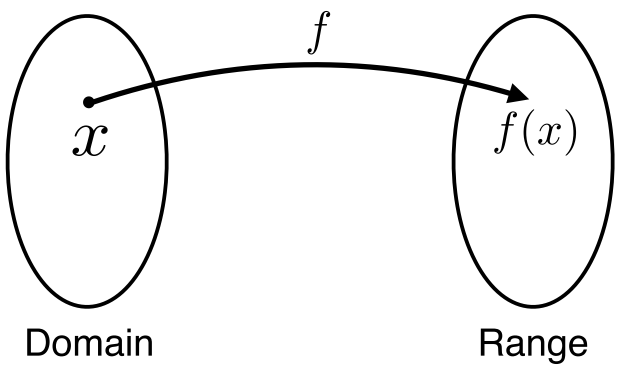
The second element is a set called the Range.



Set of Ordered Pairs

A function is an ordered triple.

The third element is a set of ordered pairs that matches elements in the domain with elements in the Range.



Set of Ordered Pairs

$$(x, f(x))$$

For each element in the Domain there is an ordered pair.

The first element in the pair is the Domain element. This is often called the “input”.

The second element of the ordered pair is an element of the Range. Its often called the “output” for that input.

A Function

Domain Range

$$f = (D, R, S)$$

$$S = \{(x, f(x)) : x \in D\}$$

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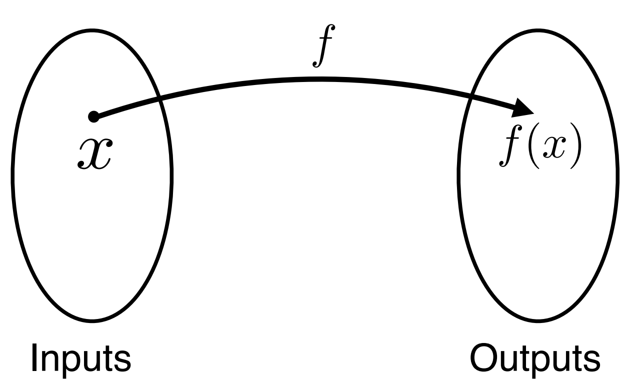
A function maps inputs to outputs. It converts x to $f(x)$.



Function Notation

$$f : D \to R$$

This is followed by "such that" and then a rule specifying $f(x)$.

Functions - Example 1

$f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x$

Functions - Example 2

Domain | Range | S

$f = (\{a,b,c,d\}, \{0,1,2,3,4,5,6,7,8\}, \{(a,0), (b,0), (c,8), (d,6)\})$

This is a completely valid function!

$f(a) = 0 \qquad f(b) = 0 \qquad f(c) = 8 \qquad f(d) = 6$

Functions - Example 3

$f : \mathbb{R} \to \{0, 1\}$ such that

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

Function - Example 4

The exponential function: $\exp$

$$\exp(-1) = 0.3679$$

$$\exp(0) = 1$$

$$\exp(1) = 2.7183$$

$$\exp(1.2) = 3.3201$$

Signals

Functions of the following types are often called “signals”:

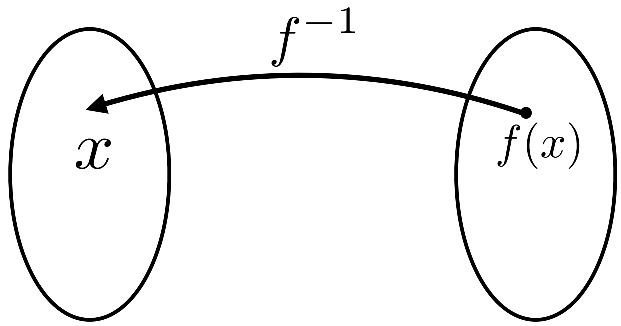
$$f : \mathbb{R} \to \mathbb{R}$$

$$f : \mathbb{R} \to \mathbb{C}$$

$$f : \mathbb{C} \to \mathbb{C}$$

Inverse Function

The inverse function converts all $f(x)$ in Outputs back to x in Inputs.



Note: Not every function has an inverse function. An invertible function is a very special thing.

Example Inverse Function

The inverse of the $\exp$ function is the $\log$ function.

$\exp(-1) = 0.3679$	$\log(0.3679) = -1$
$\exp(0) = 1$	$\log(1) = 0$
$\exp(1) = 2.7183$	$\log(2.7183) = 1$
$\exp(1.2) = 3.3201$	$\log(3.3201) = 1.2$

Transcendental Functions

$\sin$, $\cos$, and $\exp$ are very special functions

They are all continuous

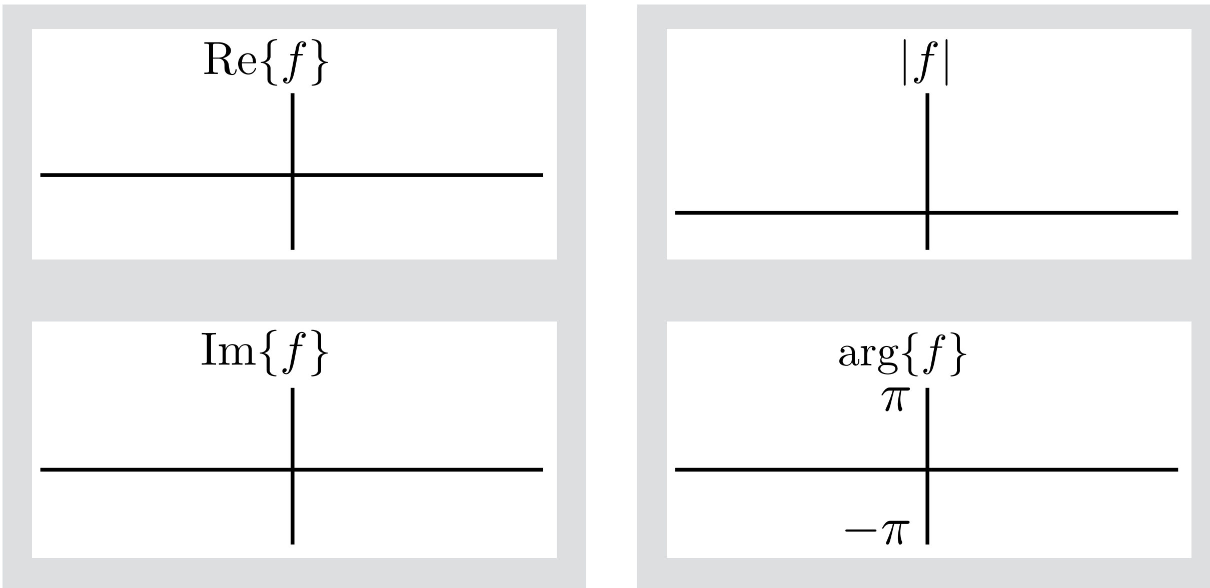
They are all infinitely differentiable

This entire class (and indeed, a huge part of engineering) is based on these functions.

Time Reversal

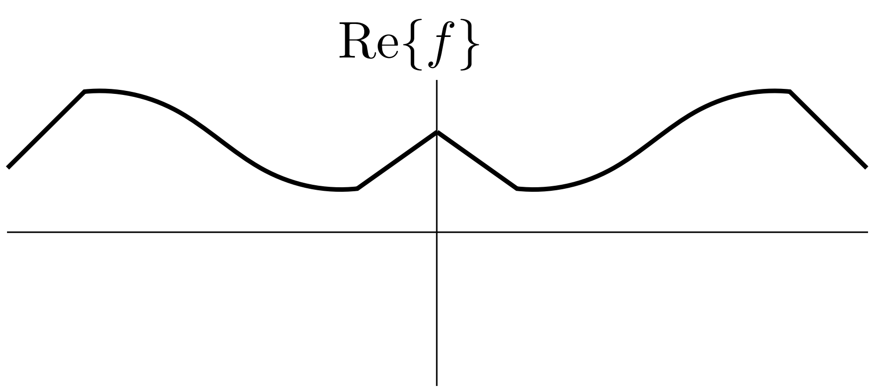
$$f^-(x) = f(-x)$$

Plotting Complex Functions



Even Functions

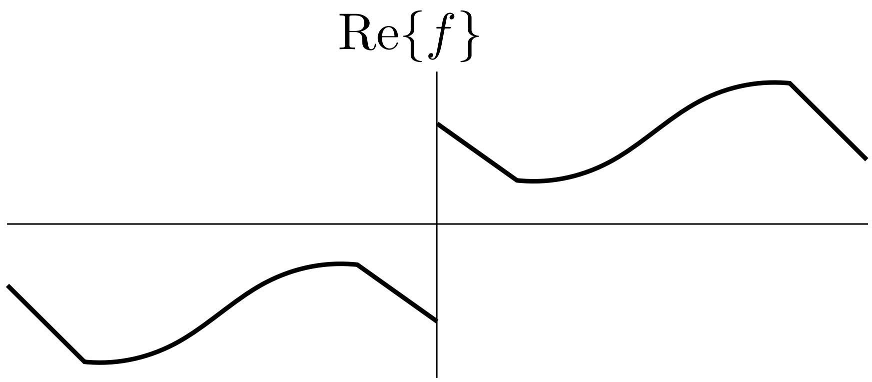
A function $f : \mathbb{R} \to \mathbb{C}$ is even means $f(x) = f^{-}(x)$ for all x .



It's a reflection about the vertical axis.

Odd Functions

A function $f : \mathbb{R} \to \mathbb{C}$ is even means $f(x) = -f^{-}(x)$ for all x.

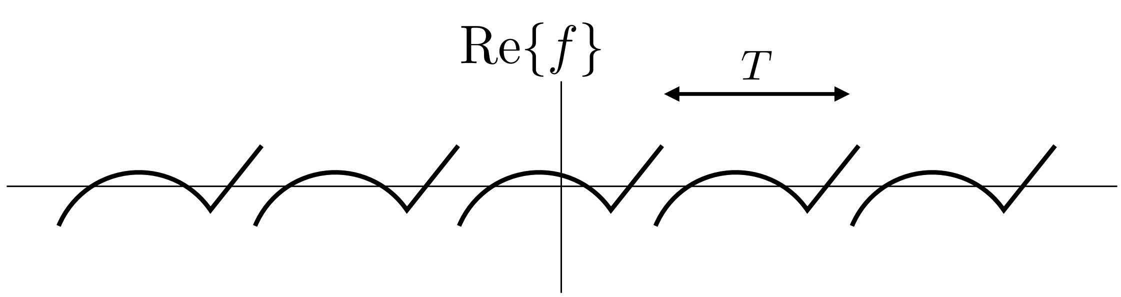


It's a rotation of half a cycle about the dotted line.

Periodic Functions

A function $f : \mathbb{R} \to \mathbb{C}$ is periodic means that there exists a number T such that $f(x + T) = f(x)$ for all x.

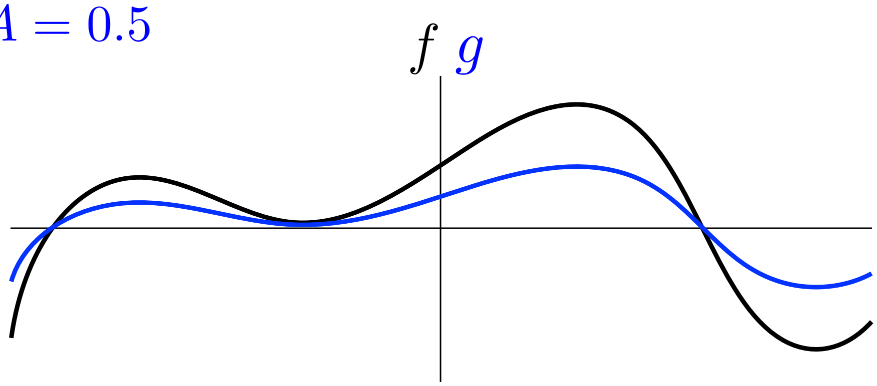
The smallest positive T for which this is true is called the Period.
It's also sometime called the fundamental period.



Scaling Function Outputs

$g(x) = A\,f(x)$

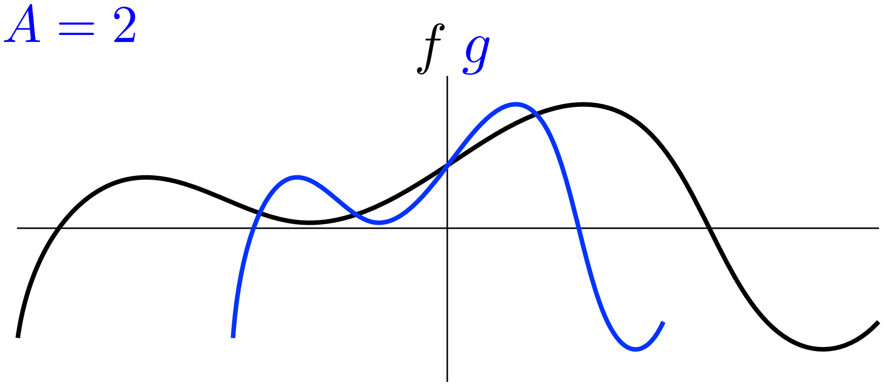
Ex: $A = 0.5$



Scaling Function Inputs

$g(x) = f(A\,x)$

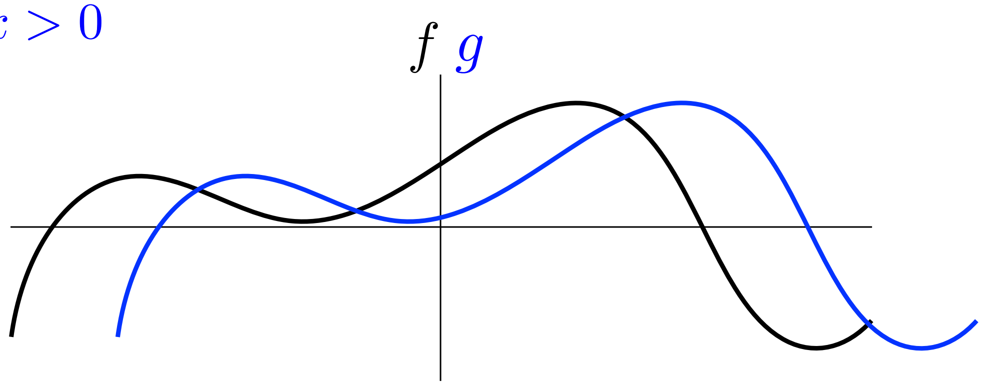
Ex: $A = 2$



Shifting Functions

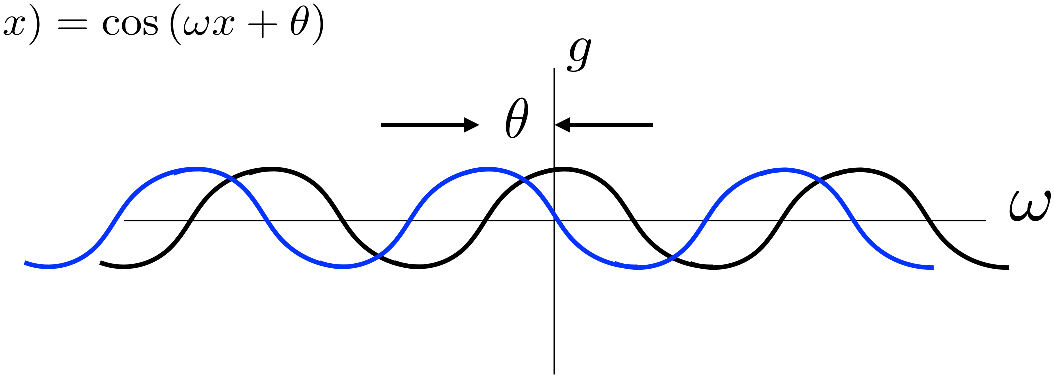
$g(x) = f(x - k)$

Ex: $k > 0$



Shifting Sinusoids

$g(x) = \cos(\omega x + \theta)$



θ is called the phase of g.

The same is true for sine.

Shifting And Scaling

$g_1(x) = f(A(x-y))$

$g_2(x) = f(Ax-y)$

This always causes confusion.
But there's a trick!

First make a new function h that only differs from f by one factor.

$h_1(x) = f(x-y)$

So h_1 is related to f by a scaling of the inputs

Now g_1 is related to h_1 by a scaling.

$g_1(x) = h_1(Ax)$

And that tells us how the function changes.
First shift and then scale!

Shifting And Scaling

Final note: it often helps to think about where the argument of the function is 0.

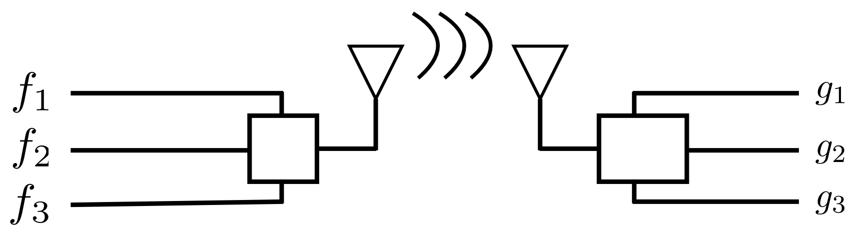
Other Types of Functions

$$f : \mathbb{C}^N \to \mathbb{C}^M$$

This function maps vectors of N complex numbers to vectors of M complex numbers.

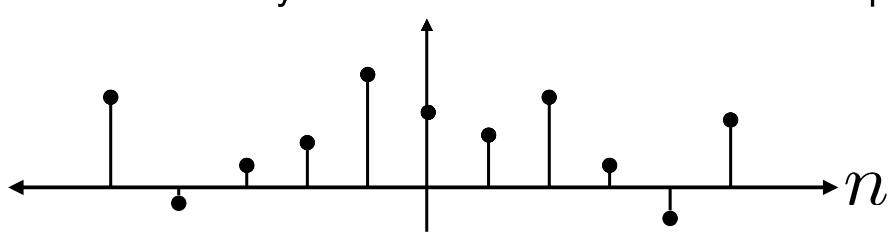
Ex: $f(x) = A\,x$ where A is a matrix

Ex:



Discrete Functions

Discrete functions are only defined at a countable number of points.



Discrete functions can (possibly) take on any value, but they are only defined at specific points.

Note: It's not the case that the function is 0 almost everywhere, and non-zero at specific points. The function *is only defined* at specific points.

Examples of Discrete Functions

Number of students in a teachers classroom v year of teaching

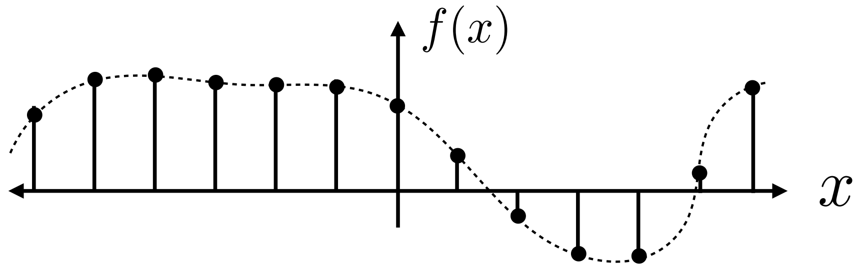
Sick feeling v number of medicine pills taken

Accident response time v number of ambulances in service

$f : \mathbb{Z} \to \mathbb{C}$ such that $f(n) = n + i\,n^2$.

Sampling

We start with a continuous function.



We extract values at specific times.

$$f_s[n] = f(n\,\Delta) \text{ for some spacing } \Delta .$$