Signal Processing and Linear Systems1

Lecture 3: Systems

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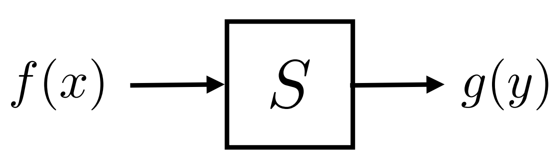
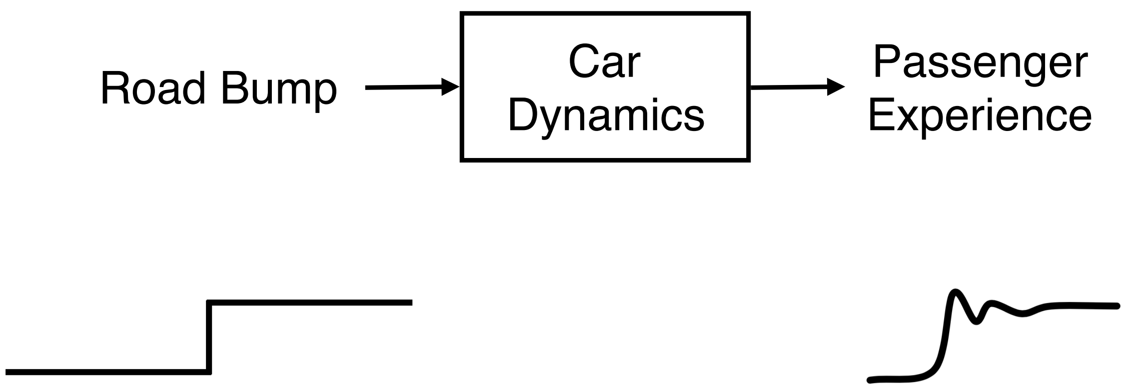
A system accepts a function and outputs a function



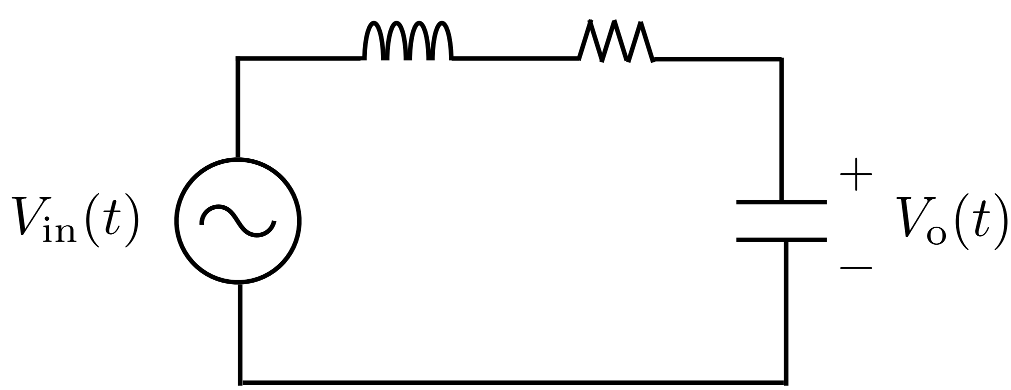
$$S\{f\} = g$$

$$S\{f(x)\}(y) = g(y)$$

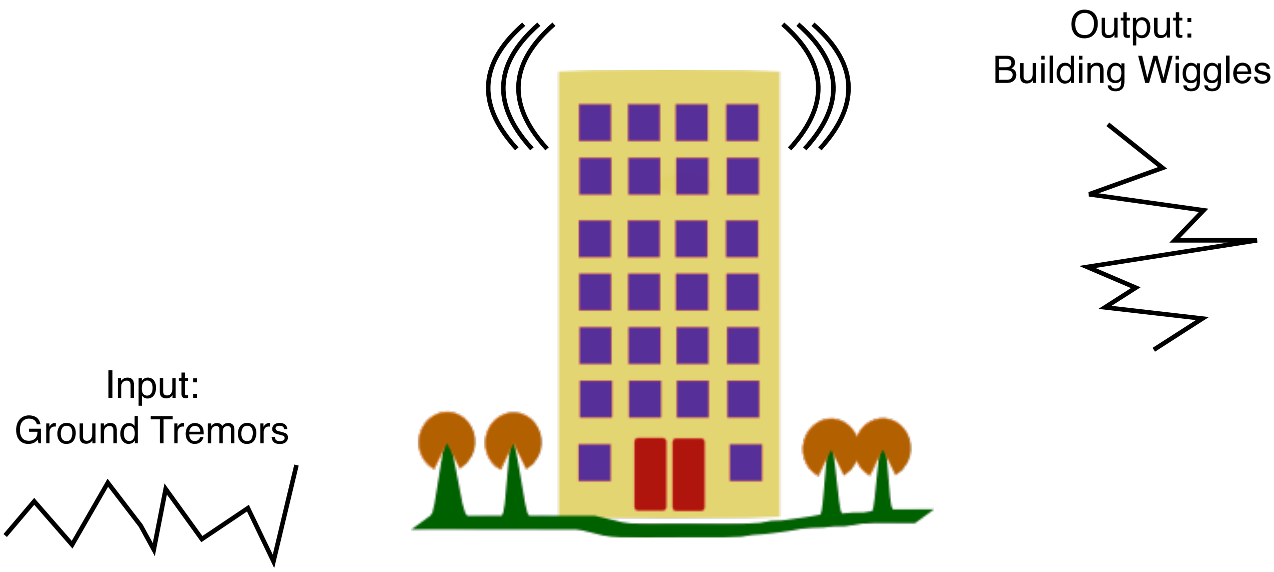
Example



Example



Example



Applications

Experiment Design

How many data points do we need to understand our system?

Which data points should we measure?

Channel Equalization

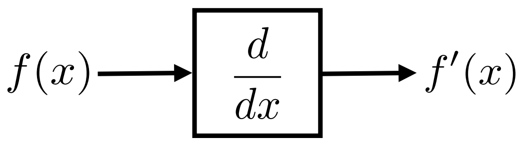
Can we undo distortions of our communications channel?

Robustification

How can we change our system (e.g. add more struts to our building) to make it more structurally sound in a cost effective way?

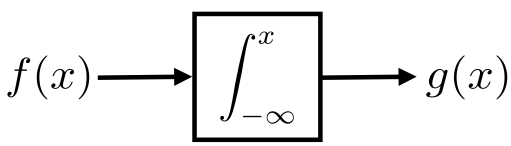
Examples

Differentiation



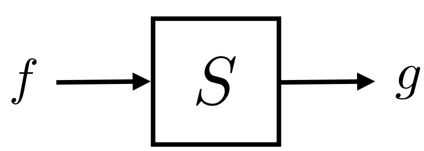
$$S\{f\} = f'$$

Integration



$$S\{f\}(x) = \int_{-\infty}^{x} f(\gamma)\, d\gamma$$

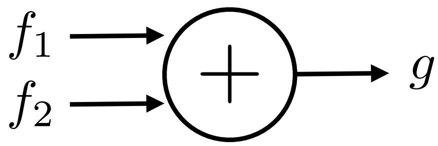
Examples



$$\begin{aligned} g(x) &= S\{f\}(x) \\ &= 4f(x-3) + \left(f'(x)\right)^2 + \cos(2\pi x)e^{i8} + 3 \end{aligned}$$

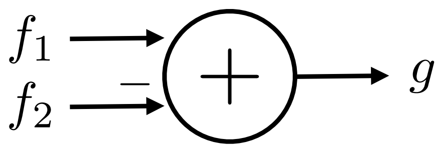
Systems Can Have Multiple Inputs

Sum



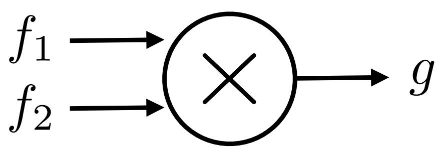
$$g = f_1 + f_2$$

Difference



$$g = f_1 - f_2$$

Multiplication



$$g = f_1 \, f_2$$

Identity System

The identity system, denoted I, is the system where

$$I\{f\} = f$$

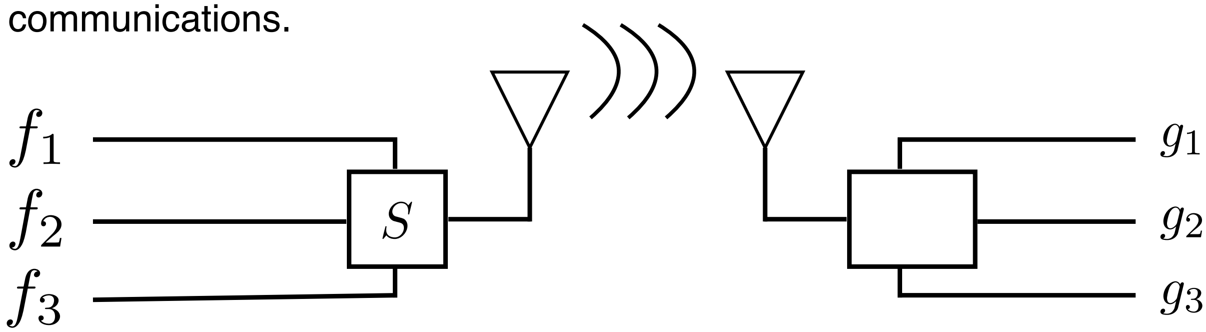
for all functions f .

Invertible

A system S is invertible means there exists a system S^{-1} such that

$S\,S^{-1} = I$ for all functions.

Invertible systems (or systems that are *almost* invertible) are required for communications.



We want $g \approx f$.

Linear Systems

A system S is linear means it satisfies both of the following

$$S\{f_1 + f_2\} = S f_1 + S f_2$$

$$S\{a\, f\} = a\, S\{f\}$$

where f is a function and a is a scalar.

Extended Linearity

For a linear system S

$$S\left\{\sum_{n=-\infty}^{\infty} a_n\, x_n\right\} = \sum_{n=-\infty}^{\infty} S\left\{a_n\, x_n\right\} = \sum_{n=-\infty}^{\infty} a_n\, S\left\{x_n\right\}$$

$$\begin{aligned} S\left\{\int_{-\infty}^{\infty} a(\gamma)\, x_\gamma(\tau)\, d\gamma\right\} &= \int_{-\infty}^{\infty} S\left\{a(\gamma)\, x_\gamma(\tau)\right\}\, d\gamma \\ &= \int_{-\infty}^{\infty} a(\gamma)\, S\left\{x_\gamma(\tau)\right\}\, d\gamma \end{aligned}$$

Trivial Input

For a linear system S

$$S\{0\} = 0$$

Proof: $$S\{0\} = S\{0\, f\} = 0\, S\{f\} = 0\ .$$

Proving a System is Linear

Ex: $S\{f\}(x) = 18\,f(x-3) + 21\,f(x+1)$

Proof: We must show that S satisfies $S\{\alpha\, f\} = \alpha\, S\{f\}$ and $S\{f+g\} = S\{f\} + S\{g\}$ for all functions f and g and scalars α.

It suffices to show $S\{\alpha\, f + \beta\, g\} = \alpha\, S\{f\} + \beta\, S\{g\}$.

$$S\{\alpha\, f + \beta\, g\}(x) = 18\,[\alpha\, f(x-3) + \beta\, g(x-3)] + 21\,[\alpha\, f(x+1) + \beta\, g(x+1)]$$

$$= \alpha\,[18\, f(x-3) + 21\, f(x+1)] + \beta\,[18\, g(x-3) + 21\, g(x+1)]$$

$$= \alpha\, S\{f\}(x) + \beta\, S\{g\}(x).$$

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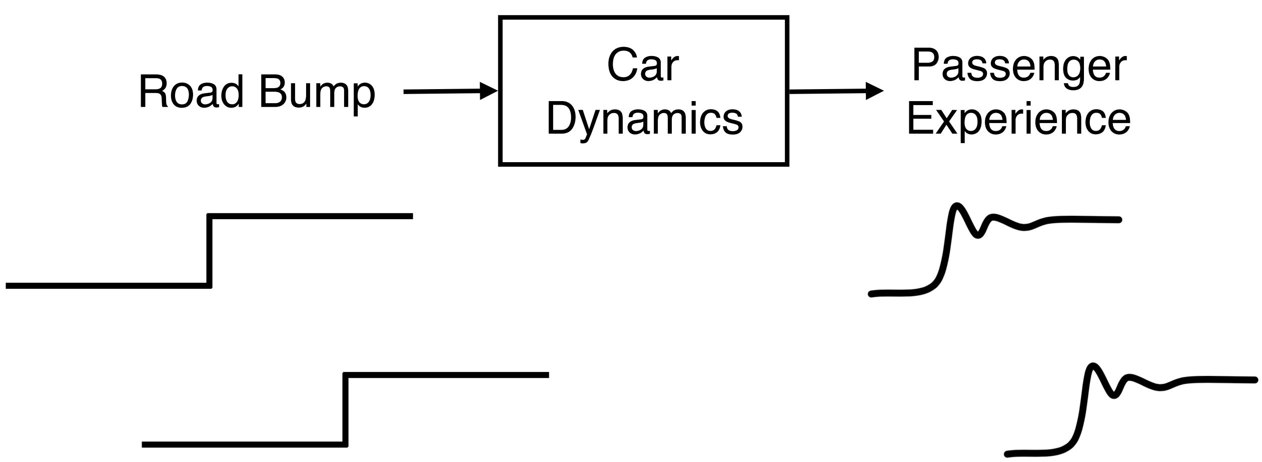
Shift Invariant

A system is shift invariant means that if you shift the input then the output shifts the same amount.



Shift Invariant

A system is shift invariant means that if you shift the input then the output shifts the same amount.

$$g(y) = S\left\{f(x)\right\}(y)$$

$$\Rightarrow \;\; g(y-\Delta) = S\left\{f(x-\Delta)\right\}(y)$$

Vector Shift Operator

We can't shift a vector over like we can with a function. So we *wrap around*.

$$\tau_p \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_M \end{bmatrix} \right\} = \begin{bmatrix} v_{M-p-1} \\ \vdots \\ v_M \\ v_1 \\ \vdots \\ v_{M-p} \end{bmatrix}$$

Vector Shift Operator

Examples:

$$\tau_1 \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} v_5 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \right\} \qquad \tau_3 \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} v_3 \\ v_4 \\ v_5 \\ v_1 \\ v_2 \end{bmatrix} \right\}$$

Circularly Shift Invariant

Consider a system $S : \mathbb{C}^M \to \mathbb{C}^N$.

S is shift invariant means that if we circularly shift the input then we circularly shift the output.

$$S\left\{\tau_p\{v\}\right\} = \tau_p\left\{S\{v\}\right\}$$

Stable

A system is Bounded Input Bounded Output stable means if

$$\sup f < \infty \quad \text{then} \quad \sup S\{f\} < \infty \ .$$

(If you don't know what the supremum is, just pretend it says maximum.)

Ex: consider an airplane system where the input is a gust of wind and the output is the altitude of the airplane.

We *REALLY* want this system to be stable!

Causal

A system is causal means it doesn't depend on future input values.

Note: it can depend on past input values.

Memory

A system has memory means the output at a given time is a function of past inputs.

A system that is not a function of a past inputs is said to be memoryless.

The output of a causal memoryless system is only a function of present time values.

Exs: Input: air conditioner strength, Output: room temperature

Input: gun orientation, Output: velocity of bullet

Input: plane aileron position, Output: plane attitude (no wind)