Signal Processing and Linear Systems1

Lecture 4: Characterizing Systems

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Our goal will be to develop a way to learn how the system behaves.

In general, this is a very difficult thing to do.

We will see that if our system has some of the nice properties we've already discussed, then this becomes remarkably easy.

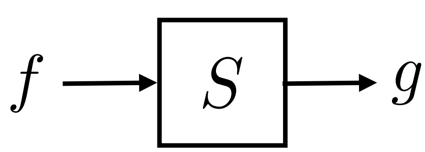
Outline

Discrete Systems

Continuous Systems

Discrete Systems

Consider a system that accepts a vector and outputs a vector.



We want to learn how this system behaves. How can we do that?

In general, we have to check every vector that we're interested in. That can take a long (perhaps infinite) amount of time.

Suppose we know that our system is linear. That helps a lot!

Consider the example where $S : \mathbb{C}^3 \to \mathbb{C}^3$

Any vector $v \in \mathbb{C}^3$ can be written as

$$v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

What is the output when our system is applied to this vector?

$$S \left\{ v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$= S \left\{ v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} + S \left\{ v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} + S \left\{ v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$= v_1 \, S \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} + v_2 \, S \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} + v_3 \, S \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Now we don't need to know the output of every vector. We only need to know the output from three vectors!!!

In general, for a linear system $S : \mathbb{C}^M \to \mathbb{C}^N$

$$S\{v\} = \sum_{i=1}^{M} v_i \, S\{e_i\}$$

This is called the superposition summation.

This shows that we only need to know the outputs for M vectors in order to know the output for any vector.

What if the system *S* is also shift invariant?

These are called Linear Shift Invariant (LSI) systems.

$$S\left\{\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}\right\} = S\left\{\tau_1\left\{\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}\right\}\right\} = \tau_1\left\{S\left\{\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}\right\}\right\}$$

In general $S\{e_i\} = S\{\tau_i\{e_1\}\} = \tau_i\{S\{e_1\}\}$

$$S\{e_i\} = S\{\tau_i\{e_1\}\} = \tau_i\{S\{e_1\}\}$$

Recall the superposition summation $S\{v\} = \sum_{i=1}^{M} v_i \, S\{e_i\}$

Let's combine these two:

$$S\{v\} = \sum_{i=1}^{M} v_i \, \tau_i \, \{S\{e_1\}\}$$

We only need to know the output for 1 vector to determine the output of any vector!!!

$$S\{v\} = \sum_{i=1}^{M} v_i \, \tau_i \, \{S\{e_1\}\}$$

e_1 is so important that we give it a special name and symbol.

$\delta = e_1$ is called the Kronecker impulse (or delta) function.

$S\{e_1\}$ is so important that we give it a special name and symbol.

$h = S\{e_1\}$ is called the impulse response.

With these symbols, $S\{v\} = \sum_{i=1}^{N} v_i \, \tau_i\{h\}$

Circular Convolution

Suppose $f, g \in \mathbb{C}^N$. Then the circular convolution of *f* and *g* is

$$f \circledast g = \sum_{i=1}^{N} f_i \, \tau_i\{g\}$$

$$(f \circledast g)[n] = \sum_{m=1}^{N} f[m] \, g\left[(n-m)_{\mathrm{mod}\, N}\right]$$

This shows that for an LSI system

$$S\{v\} = \sum_{i=1}^{N} v_i \, \tau_i\{h\} = v \circledast h.$$

<u>Major Theorem</u>:

For an discrete LSI system, the output is equal to the input circularly convolved with the impulse response.

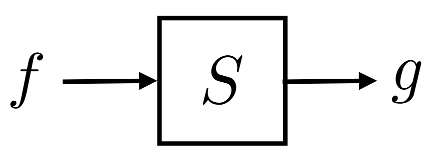
Outline

Discrete Systems

Continuous Systems

Continuous Systems

Consider a system that accepts a function and outputs a function.



We want to learn how this system behaves. How can we do that?

In general, we have to check every function that we're interested in.
That can take a long (perhaps infinite) amount of time.

Suppose we know that our system is linear. That helps a lot!

But first, we need a new tool.

With discrete systems we had a vector δ such that

$$\delta \cdot x = \sum_{i=1}^{N} \delta_i \, x_i = x_1$$

With continuous systems, we need an analogous function

$$\int_{-\infty}^{\infty} \delta(s) \, x(s) ds = x(0)$$

Impulse Function

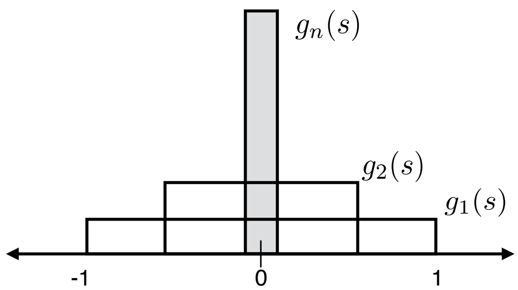
The Dirac Delta Function δ is the function that satisfies the following:

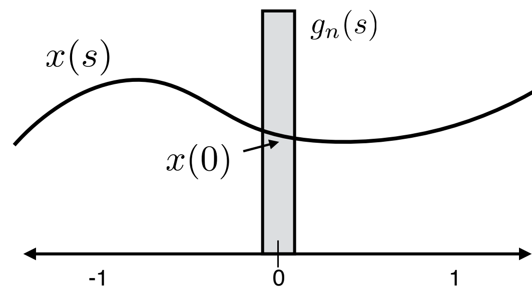
$$\int_{-\infty}^{\infty} \delta(s) \, x(s) ds = x(0)$$

Basic idea: δ acts over a time interval which is very small. During this small time interval, $x(s) \approx x(0)$.

Impulse Function Intuition

Approximate $\delta(s)$ as $g_n(s) = n\,\Pi(ns)$. The area of g_n is $n\,(1/n) = 1$.





$$\int_{-\infty}^{\infty} x(s)\,\delta(s)\,ds = \lim_{n\to\infty} \int_{-\infty}^{\infty} x(s)\,g_n(s)\,ds = x(0) \int_{-\infty}^{\infty} g_n(s)\,ds = x(0)$$

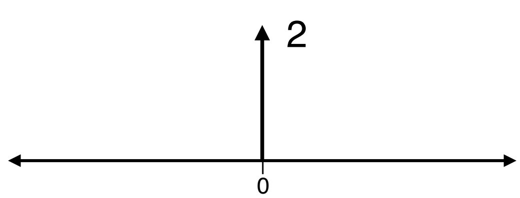
Scaled Impulses

$\alpha\,\delta$ is an impulse at time 0 with magnitude or strength or area α .

$$\int_{-\infty}^{\infty} \alpha\,\delta(s)\,x(s)ds = \alpha\,x(0)$$

On plots, draw an arrow and write the strength next to the arrow.

Ex: 2δ



Scaling the Argument of Impulses

$$\delta(as) = \frac{1}{|a|}\,\delta(s)$$

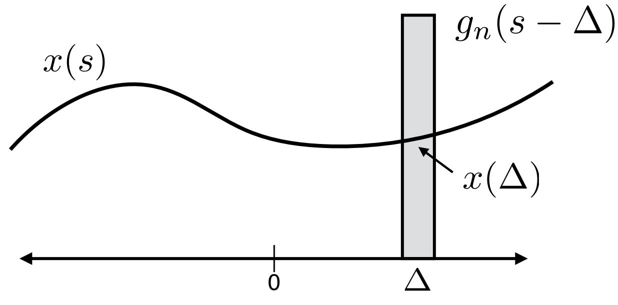
Again, this can be proved with variable substitution.

Sifting Property

When the impulse function is shifted, we get a shifted value of x.

$$\int_{-\infty}^{\infty} x(s)\,\delta(s-\Delta)\,ds = x(\Delta)$$

This can be proved by a variable substitution.



Now that we have the impulse function, how can we characterize a continuous linear system?

Before, we noted that any vector $v \in \mathbb{C}^3$ could be written as

$$v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We need to write a function x similarly. It turns out, now we can!

$$x(\gamma) = \int_{-\infty}^{\infty} x(s)\delta(\gamma - s)\, ds$$

$$\begin{aligned} S\{x(\gamma)\} &= S\left\{\int_{-\infty}^{\infty} x(s)\delta(\gamma - s)\, ds\right\} \\ &= \int_{-\infty}^{\infty} S\{x(s)\delta(\gamma - s)\}\; ds \\ &= \int_{-\infty}^{\infty} x(s)\, S\{\delta(\gamma - s)\}\; ds \\ &= \int_{-\infty}^{\infty} x(s)\, h_\gamma(s)\, ds \end{aligned}$$

where h_γ is the response of the system applied to an impulse at location.

This is called the Superposition integral.

What if the system is also shift invariant?

$$S\{x(\gamma)\} = \int_{-\infty}^{\infty} x(s)\, S\left\{\delta(\gamma - s)\right\}\, ds$$
$$= \int_{-\infty}^{\infty} x(s)\, h(\gamma - s)\, ds$$

where h is the response of the system to an impulse function (the impulse response).

To determine the output of an LSI system to any function, we only need to know how the system responds to one function - an impulse!!!

Convolution

The convolution of *f* and *g* is

$$(f * g)(\gamma) = \int_{-\infty}^{\infty} f(s)\, g(\gamma - s)\, ds$$

With this notation,

$$S\{x\} = x * h$$

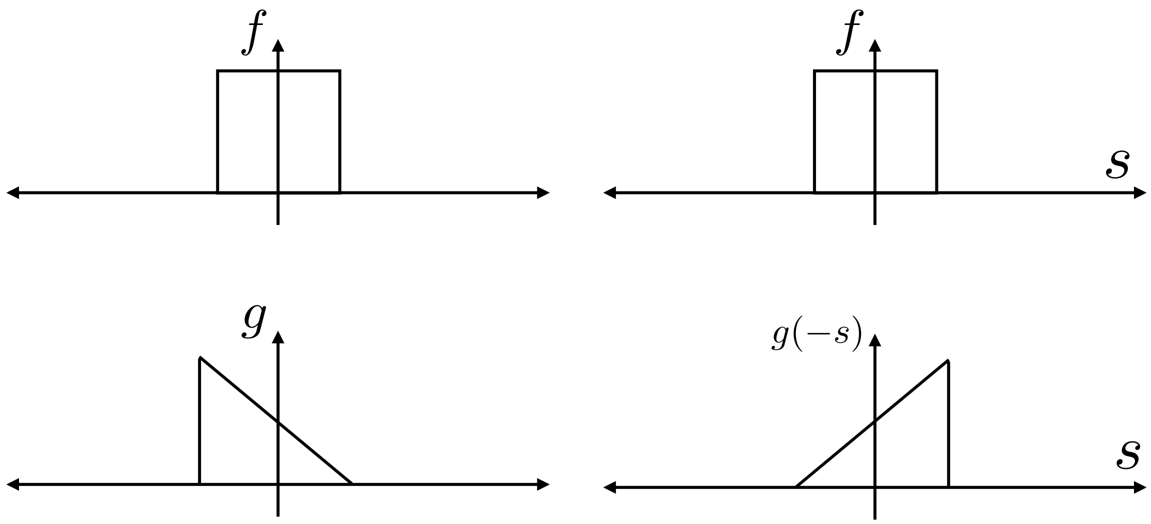
<u>Major theorem</u>:

For a continuous LSI system, the output is equal to the input convolved with the impulse response.

Graphical Interpretation

$$(f * g)(\gamma) = \int_{-\infty}^{\infty} f(s)\, g(\gamma - s)\, ds$$

Let's consider a specific γ.



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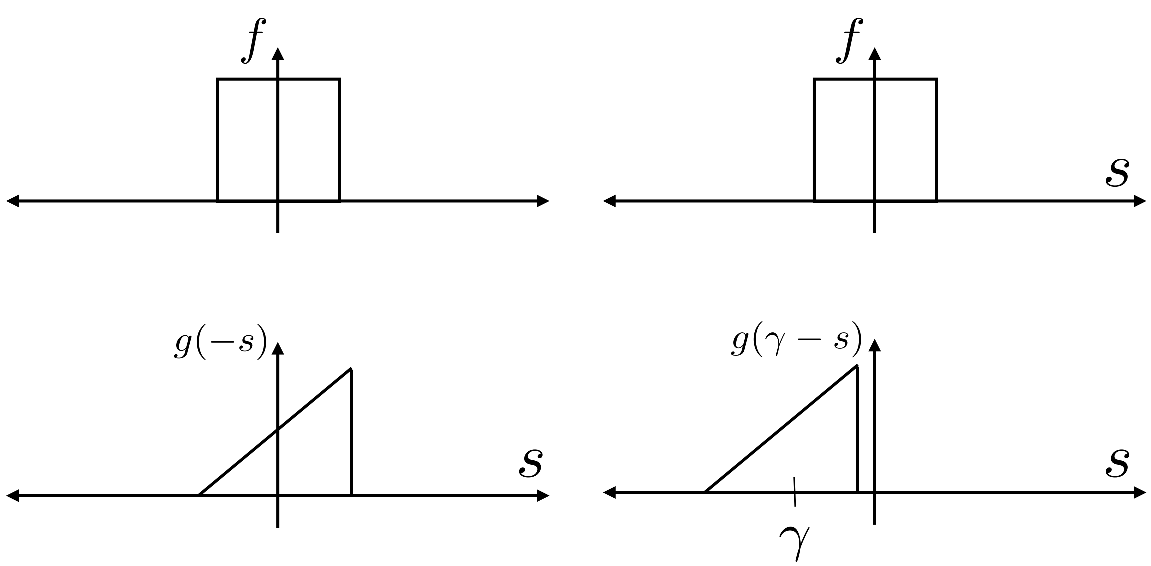

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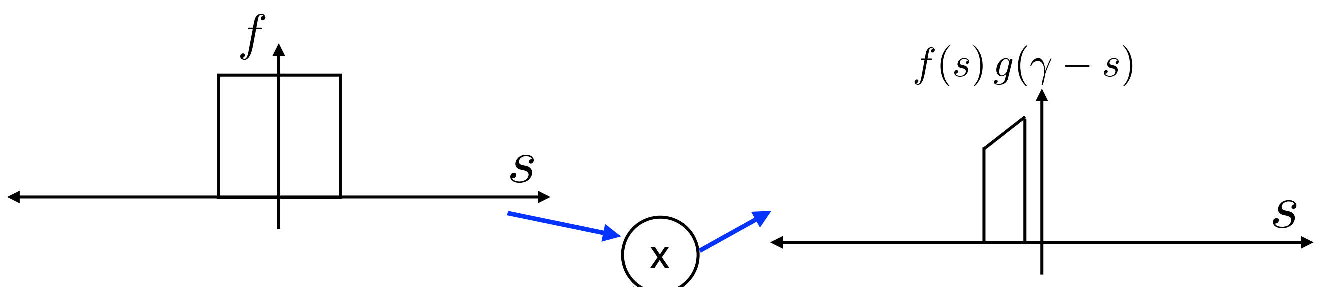
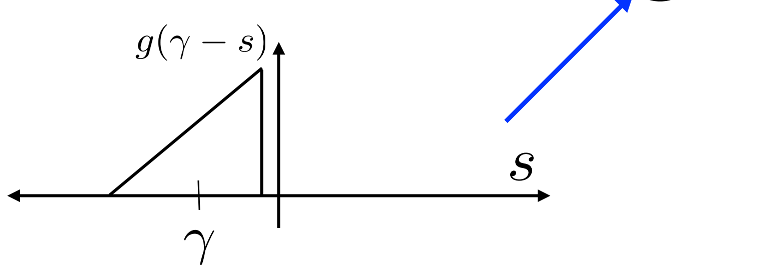


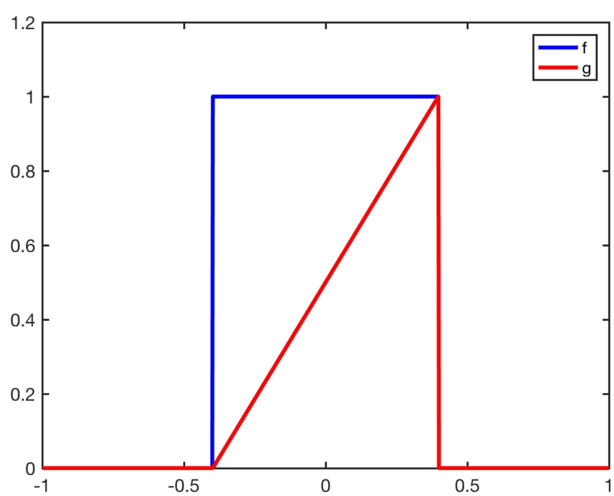


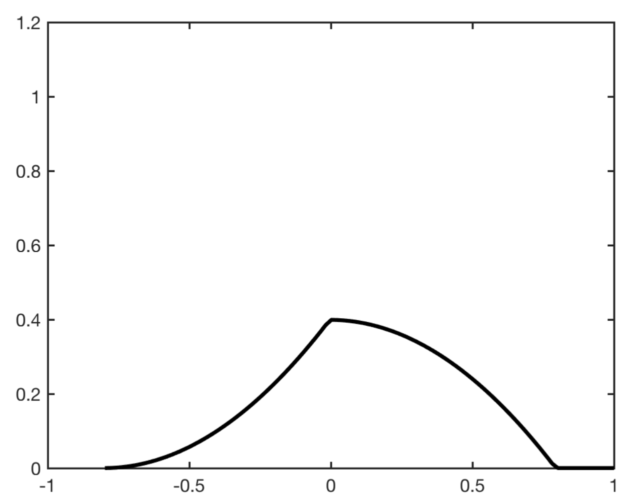
And then we integrate to get the area under the curve.

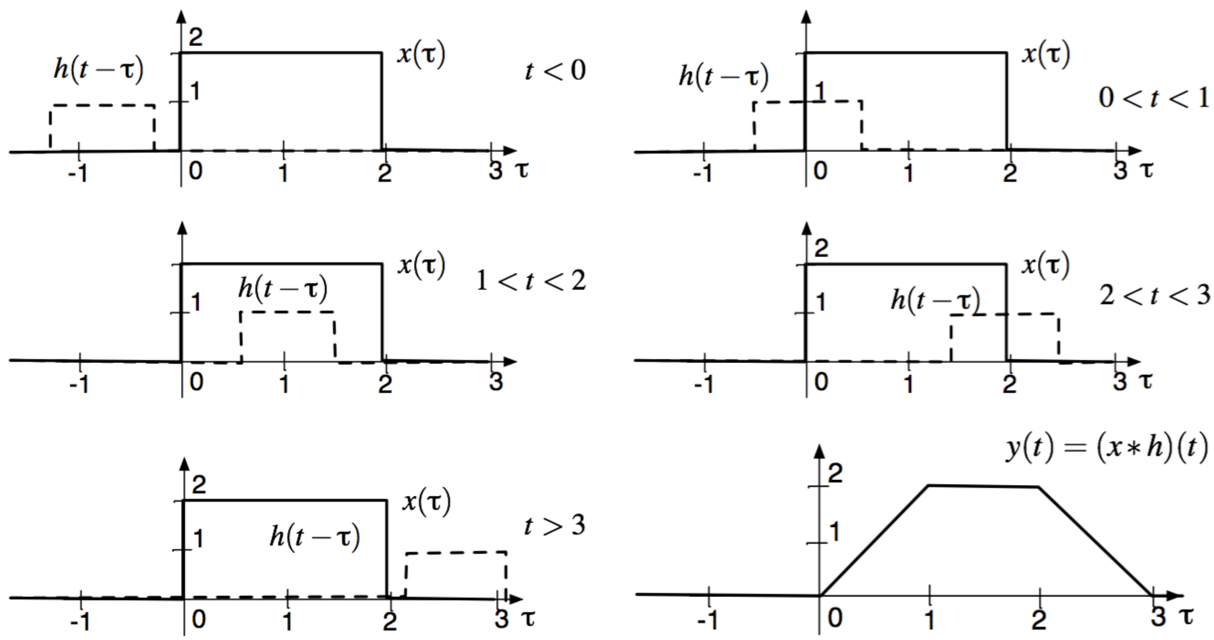
That gives us the value of

$$(f * g)(\gamma)$$

"Flip and Slide"



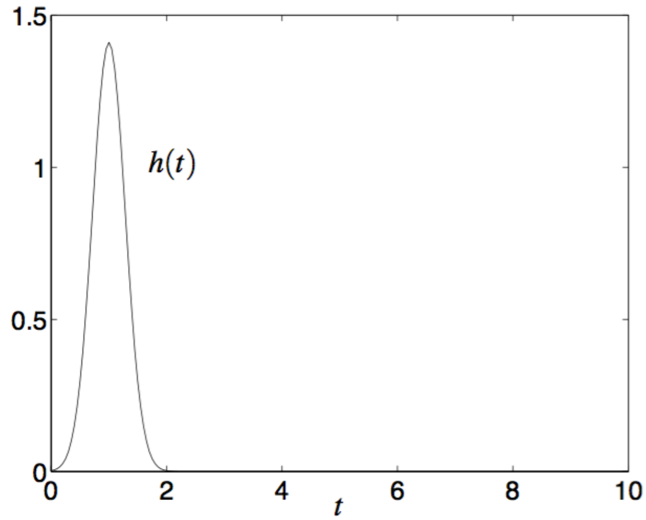




Examples of System Responses

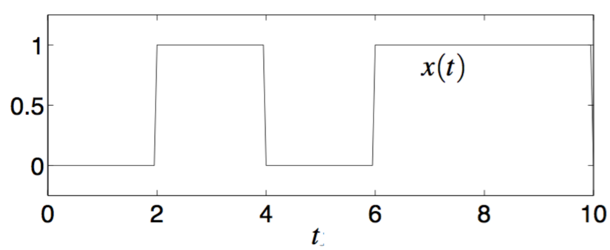
Twisted Pair Cable

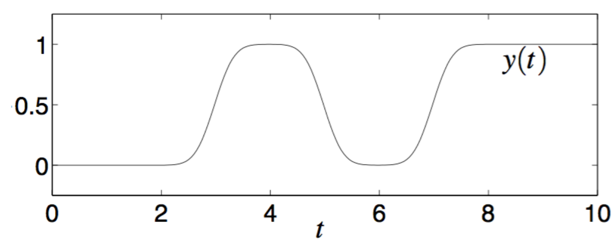
The output is delayed and smoothed.



Transmitting 0.5 bits / sec.

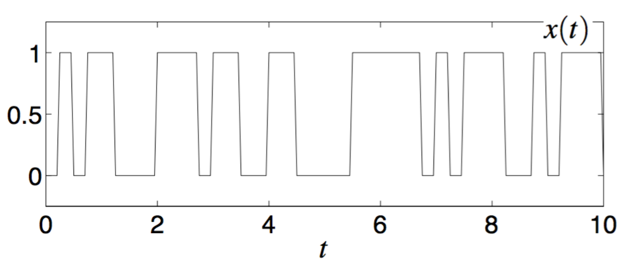
We can easily see the 1s and 0s in the output.

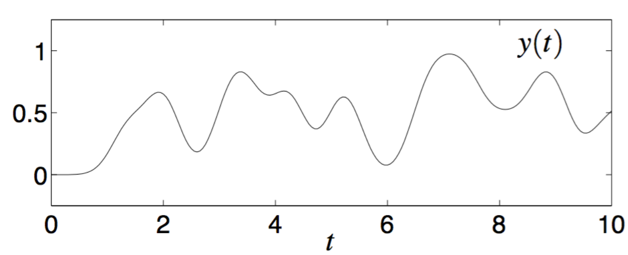




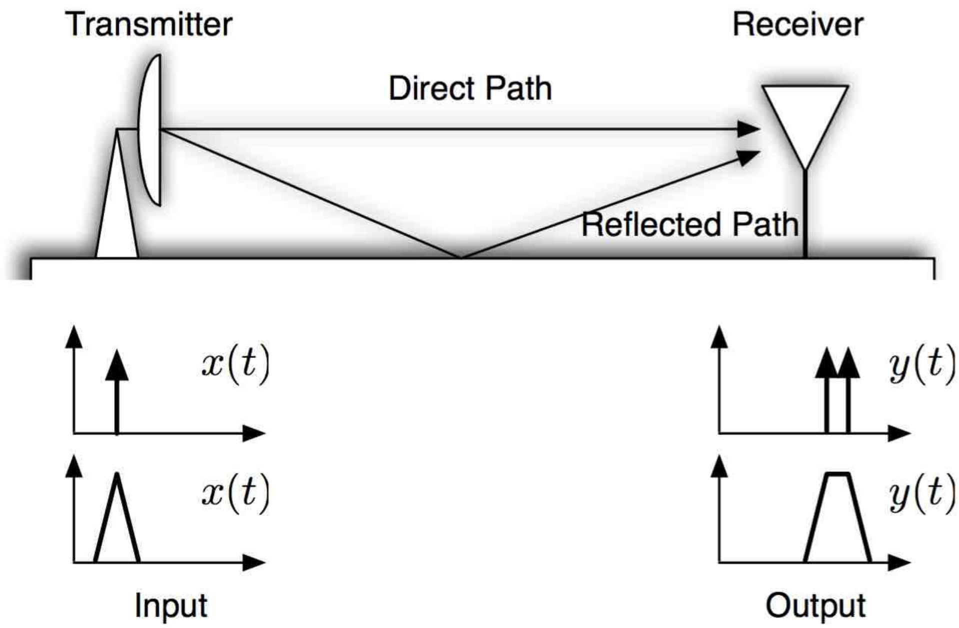
Transmitting 4 bits / sec.

The signal appears lost.



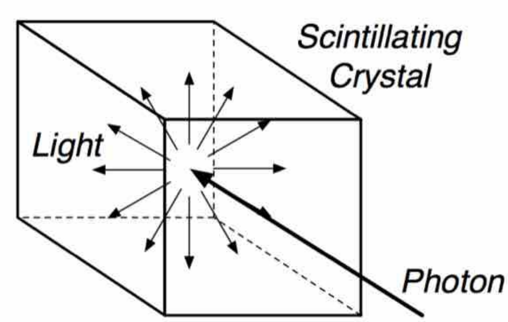


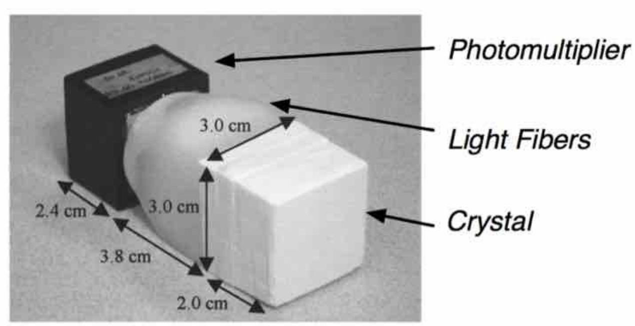
Multi-path Echoes



High Energy Photon Detectors

Can be modeled as having a simple exponential decay impulse response.



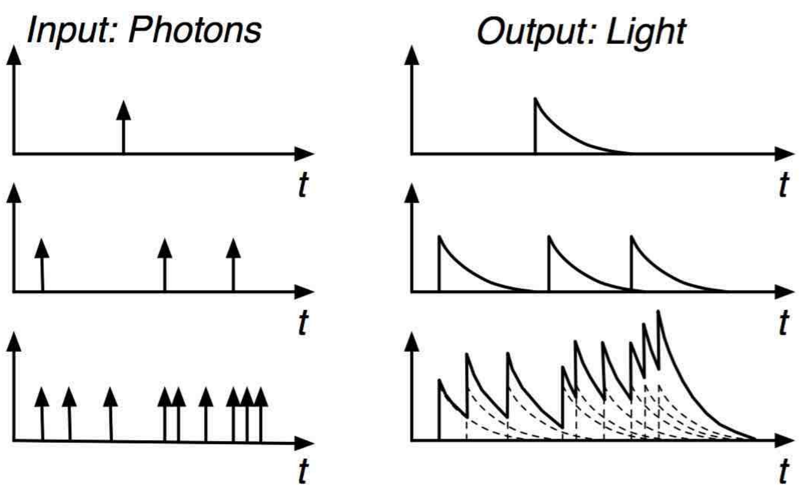


From: Doshi et al, Med Phys. 27(7), p1535 July 2000

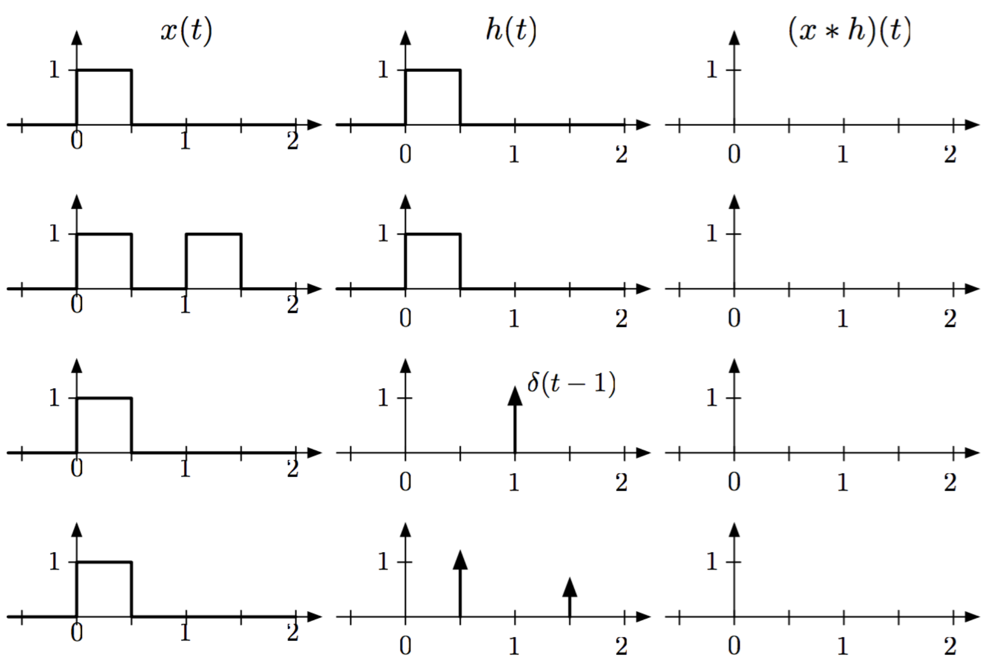
They are used in Positron Emission Tomography (PET) and Nuclear Scintigraphy medical imaging modalities.

Input: a series of photons (modeled as impulses)

Output: recorded light is a superposition of impulse responses.



Try these:



Properties of Convolution

Associative $$f * (g * h) = (f * g) * h$$

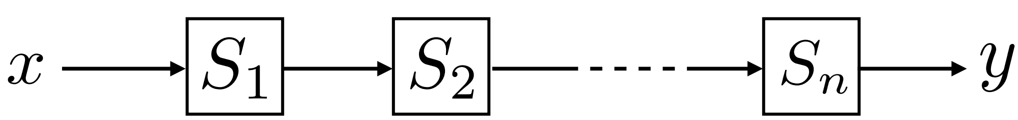
Commutative $$f * g = g * f$$

Distributive $$f * (g + h) = (f * g) + (f * h)$$

Identity $$f * \delta = f$$

Multiple Systems

What if we had multiple systems?



Even if we knew the impulse responses, to estimate the output for a given input, we would have to perform *n* convolutions!!!

That would be hard and boring.

There’s a better way…