Signal Processing and Linear Systems1

Lecture 5: Fourier Series and Fourier Transform

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Now we know that we can write any function as a linear combination of impulse functions.

$$f(x) = \int_{-\infty}^{\infty} f(s)\,\delta(x-s)\,ds$$

But impulses are difficult to work with. What is the response, for example, to a system that squares the input?

The above formulation let us determine interesting properties of LSI systems, but it doesn't really help us determine the impulse response of a system.

Wouldn't it be nice if we could write x as a linear combination of extremely nice functions?

The nicest functions we have are sines and cosines.

We can evaluate them anywhere, they're periodic, and we can differentiate and integrate them as many times as we want.

Let's consider a function $f : [0, P] \to \mathbb{C}$.

For a moment, let's pretend that we can write a function as a linear combination of sines and cosines.

$$f(x) = \sum_{n=-\infty}^{\infty} F_n \exp\left(i\,2\pi\,\frac{nx}{P}\right)$$

If it's true that we can write f this way, then what are the values of F_n?

$$f(x) = \sum_{n=-\infty}^{\infty} F_n \exp\left(i\,2\pi\,\frac{nx}{P}\right)$$

The trick: compute the following expression

$$\frac{1}{P}\int_0^P f(x)\exp\left(-i\,2\pi\frac{mx}{P}\right)dx = \frac{1}{P}\int_0^P \sum_{n=-\infty}^{\infty} F_n \exp\left(i\,2\pi\frac{nx}{P}\right)\exp\left(-i\,2\pi\frac{mx}{P}\right)dx$$

$$= \sum_{n=-\infty}^{\infty} F_n \frac{1}{P}\int_0^P \exp\left(i\,2\pi\,\frac{n\,x}{P}\right)\exp\left(-i\,2\pi\,\frac{m\,x}{P}\right)dx = F_m \text{ !!!}$$

Now we can find all the F values.

Fourier Series

Forward Transform (Analysis Equations):

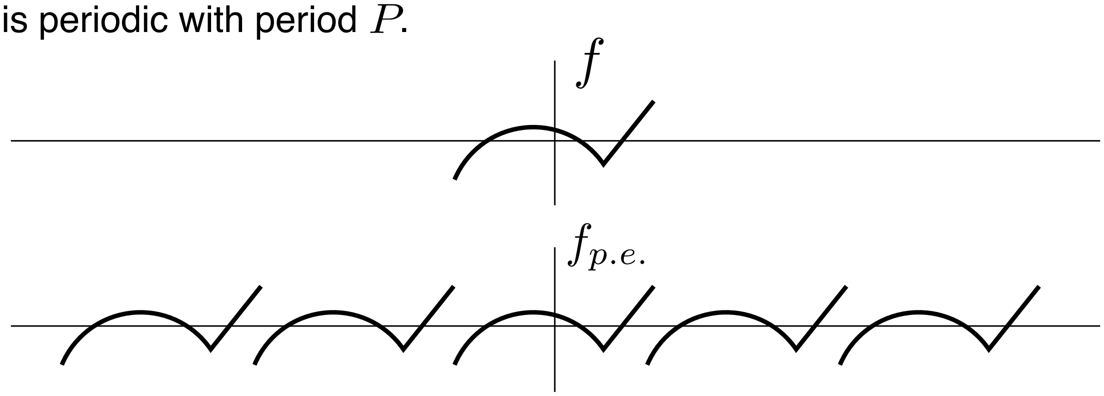
$$F_n = \frac{1}{P} \int_0^P f(x)\, \exp\left(-i\, 2\pi \frac{nx}{P}\right) dx$$

Inverse Transform (Synthesis Equations):

$$f(x) = \sum_{n=-\infty}^{\infty} F_n\, \exp\left(i\, 2\pi \frac{nx}{P}\right)$$

Periodic Extension

The periodic extension of a function $f : [0, P] \to \mathbb{C}$ is the function $f_{p.e.} : \mathbb{R} \to \mathbb{C}$ such that $f_{p.e.}(x) = f(x)$ for all $x \in [0, P)$ and $f_{p.e.}$ is periodic with period P.



If we have f, we can get $f_{p.e.}$. If we have $f_{p.e.}$, then we can get f .

More than that, adding two functions defined on [0,P) and determining the periodic extension is the same as computing the periodic extensions and adding.

The same is true of scalar multiplication.

Technically, we say that functions defined on [0,P) and periodic extensions of those functions are *isomorphic* to each other.

You don't need to know this word; it just means that we don't add or lose information by using the original function or the periodic extensions.

For any periodic function, we can compute the Fourier Series of any period. And then we can get back the original function!

Now we know that *if* we can represent a function as a sum of sines and cosines, then we can find the linear coefficients.

But how often can a periodic function be represented this way?

ALMOST ALWAYS!!!

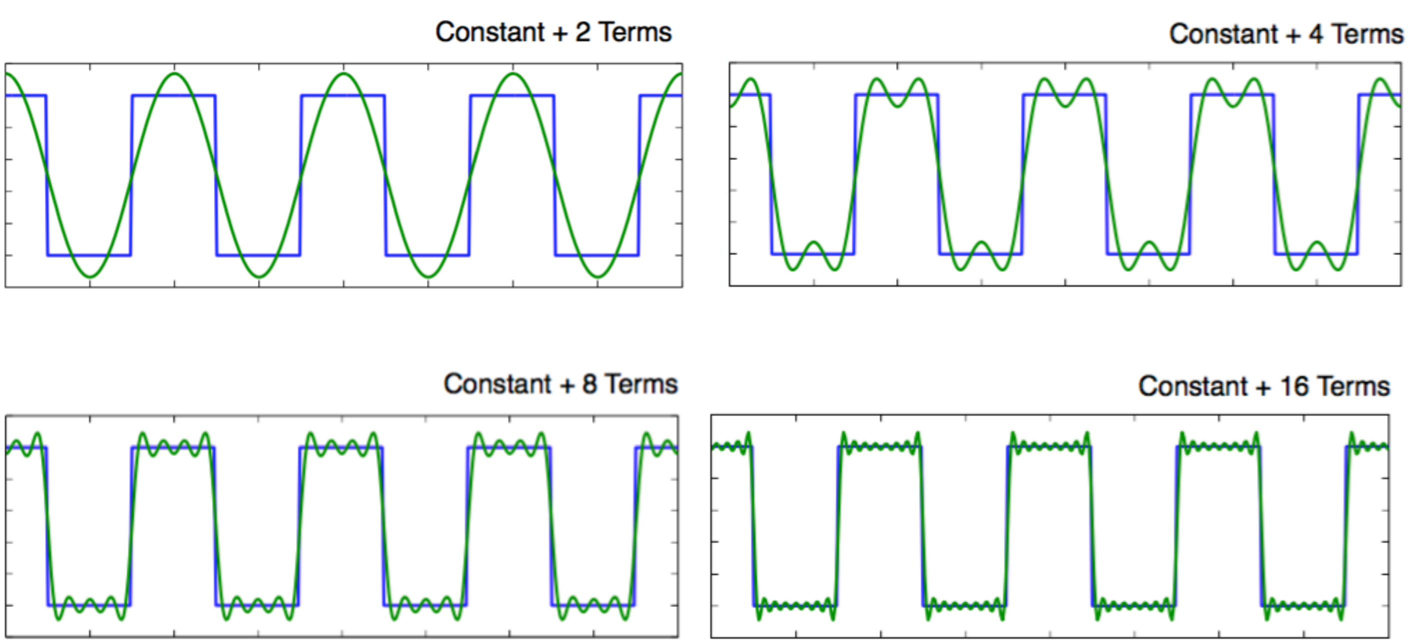
Sufficient condition: any power function can be represented as a Fourier Series.

Necessary and sufficient condition: Dirichlet conditions (see Signal Analysis) by Papoulis for details.

$$f(x) = \sum_{n=-\infty}^{\infty} F_n \, \exp\left(i \, 2\pi \frac{nx}{P}\right)$$

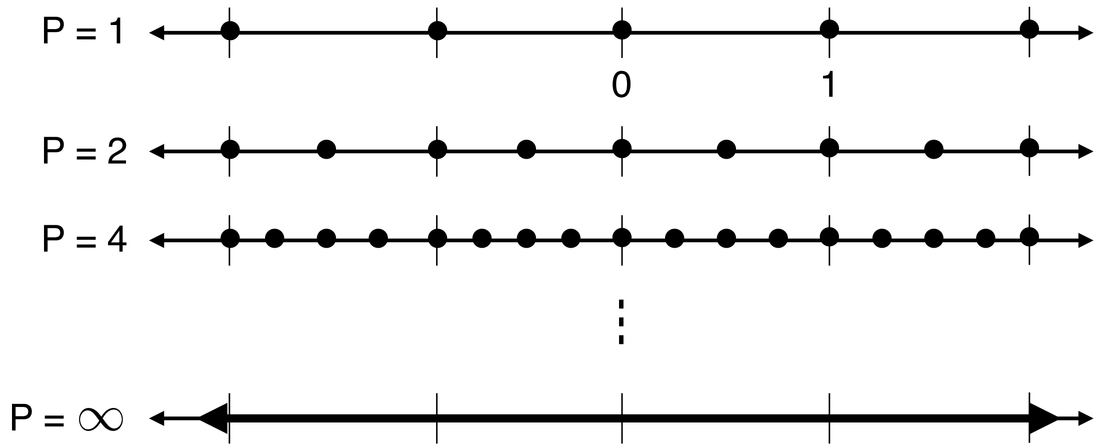
If f is real and even then $f(x) = \sum_{n=-\infty}^{\infty} F_n \cos\left(2\pi \frac{nx}{P}\right)$.

We are approximating a function as a sum of cosines with increasing frequency.



What frequencies are we using? $f(x) = \sum_{n=-\infty}^{\infty} F_n \exp\left(-i\,2\pi\frac{nx}{P}\right)$

Let $k = \dfrac{n}{P}$. Then k is the frequency in Hertz.



Recall the synthesis equation of the Fourier Series:

$$f(x) = \sum_{n=-\infty}^{\infty} F_n \exp\left(i\,2\pi\frac{nx}{P}\right)$$

The analogous equation for a function define on all real numbers is

$$f(x) = \int_{-\infty}^{\infty} F(k) \exp(i\,2\pi\, kx) dk$$

This is the synthesis equation for the ***Fourier Transform***.

Fourier Transform

Forward Transform (Analysis Equations):

$$F(k) = \mathcal{F}\{f\}(k) = \int_{-\infty}^{\infty} f(x) \exp\left(-i\, 2\pi\, kx\right) dx$$

Inverse Transform (Synthesis Equations):

$$f(x) = \mathcal{F}^{-1}\{F\}(x) = \int_{-\infty}^{\infty} F(k) \exp\left(i\, 2\pi\, kx\right) dk$$

F is called the spectrum of f.

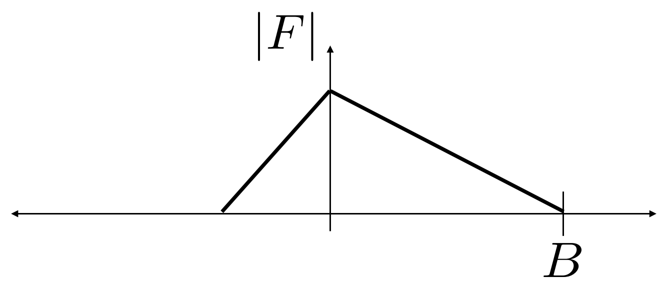
Power Spectral Density (PSD)

$$PSD\{f\}(k) = |F(k)|^2 = F(k)\, \overline{F}(k)$$

Bandwidth

The bandwidth of a function f is the smallest value B such that

$F(k) = 0$ for all $|k| > B$.



System Response

Now we know that we can represent functions as linear combinations of sines and cosines.

We will now see how to use this fact to determine a linear system's response.

But first we need one more very important tool: The Convolution Theorem.

Shift Theorem

$$\mathcal{F}\{f(x-\Delta)\}(k) = e^{-i\,2\pi\,k\Delta}\mathcal{F}\{f\}(k)$$

Proof:

$$\mathcal{F}\left\{f(x-\Delta)\right\}(k) = \int_{-\infty}^{\infty} f(x-\Delta)e^{-i2\pi kx}dx.$$

Let $u = x - \Delta$.

$$\mathcal{F}\left\{f(x-\Delta)\right\}(k) = \int_{-\infty}^{\infty} f(u)e^{-i2\pi k(u+\Delta)}dx$$

$$= e^{-i2\pi k\Delta}\int_{-\infty}^{\infty} f(u)e^{-i2\pi ku}dx. \quad \blacksquare$$

Convolution Theorem

$$\mathcal{F}\left\{f * g\right\} = \mathcal{F}\{f\}\,\mathcal{F}\{g\}$$

Proof:

$$\mathcal{F}\left\{f * g\right\} = \int_{-\infty}^{\infty}(f*g)(x)\,e^{-i2\pi kx}dx \; = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\gamma)g(x-\gamma)d\gamma\right)\,e^{-i2\pi kx}dx$$

$$= \int_{-\infty}^{\infty} f(\gamma)\left(\int_{-\infty}^{\infty} g(x-\gamma)e^{-i2\pi kx}dx\right)d\gamma \; = \int_{-\infty}^{\infty} f(\gamma)\left(e^{-i2\pi k\gamma}\right)G(k)d\gamma$$

$$= G(k)\int_{-\infty}^{\infty} f(\gamma)\left(e^{-i2\pi k\gamma}\right)d\gamma \;\; = F(k)\,G(k) \quad \blacksquare$$

The Fourier Transform Converts Convolution into Multiplication!!!

Recall that for a LSI system, $y(\gamma) = S\{f\}(\gamma) = (f * h)(\gamma).$

By the convolution theorem, $Y(k) = F(k)\, H(k).$

Now we can find the impulse response of the system!

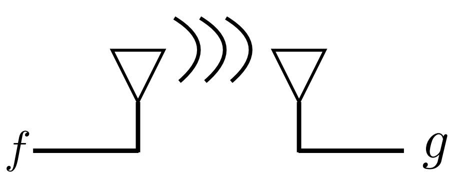
Send in a signal with the frequencies you're interested in, and use the outputs to find H.

$$H(k) = Y(k)/F(k) \qquad\qquad h = \mathcal{F}^{-1}\{H\}$$

H is called the Transfer function. $|H|$ is called the Modulation Transfer function.

Channel Equalization (Deconvolution)

Suppose we are transmitting a signal through a medium. The medium distorts the signal, and we would like to undo that distortion.



Send a known signal with the spectrum you're interested in. Record the output.

Compute $H = G/F$.

For the next signal,

1) Compute the Fourier Transform $\quad G \approx FH$
2) Divide by *H* $\quad \hat{F} = G/H \approx F$
3) Inverse Fourier Transform! $\quad \hat{f} = \mathcal{F}^{-1}\{\hat{F}\} \approx f$

What we’ve developed is so powerful that we’ll now spend a great deal of time learning about the Fourier Transform.