Signal Processing and Linear Systems1

Lecture 6: Properties of the Fourier Transform

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Fourier Transform

Forward Transform (Analysis Equations):

$$F(k) = \mathcal{F}\{f\}(k) = \int_{-\infty}^{\infty} f(x) \exp\left(-i\, 2\pi\, kx\right) dx$$

Inverse Transform (Synthesis Equations):

$$f(x) = \mathcal{F}^{-1}\{F\}(x) = \int_{-\infty}^{\infty} F(k) \exp\left(i\, 2\pi\, kx\right) dk$$

Dirac Delta Function

$$\mathcal{F}\{\delta\} = 1$$

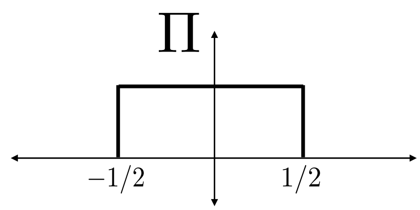
Proof:

$$\mathcal{F}\{\delta\} = \int_{-\infty}^{\infty} \delta(x) e^{-i\,2\pi\,kx} dx \;= e^{-i\,2\pi\,k\,0} \;= 1$$

$\blacksquare$

Rect Function

$$\Pi(x) = \begin{cases} 1 & |x| < 1/2 \\ 1/2 & |x| = 1/2 \\ 0 & |x| > 1/2 \end{cases}$$



$$\mathcal{F}\left\{\Pi\right\}(k) = \int_{-\infty}^{\infty} \Pi(x) e^{-i\,2\pi\,kx} dx \;= \int_{-1/2}^{1/2} e^{-i\,2\pi\,kx} dx$$

$$= \int_{-1/2}^{1/2} \cos(2\pi\,kx) - i\sin(2\pi\,kx) dx \;= \int_{-1/2}^{1/2} \cos(2\pi\,kx) dx$$

$$= \left.\frac{\sin(2\pi\,kx)}{2\pi\,k}\right|_{-1/2}^{1/2} \;= \frac{\sin(\pi\,k)}{\pi\,k} \;= \mathrm{sinc}(k)$$

Gaussian Function

$$f(x) = e^{-\pi\, x^2} \qquad \mathcal{F}\{f\}(k) = e^{-\pi\, k^2}$$

The Fourier Transform of a Gaussian is itself.

Proof:

$$\mathcal{F}(k) = \hat{f}(k) = \int_{-\infty}^{\infty} e^{-\pi x^2}\, e^{-i2\pi kx} dx$$

Now differentiate both sides with respect to *k*.

$$\hat{f}'(k) = \int_{-\infty}^{\infty} e^{-\pi x^2}\, (-2\pi x) e^{-i2\pi kx} dx$$

Integrate by parts where $u(x) = e^{-i2\pi kx}$, and $v'(x) = e^{-\pi x^2}(-2\pi x)$.

$$\hat{f}'(k) = -2\pi k \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-i2\pi kx} dx = (-2\pi k)\hat{f}(k)$$

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$$\hat{f}'(k) = -2\pi k \int_{-\infty}^{\infty} e^{-\pi x^t} e^{-i2\pi kx} dx = (-2\pi k)\hat{f}(k)$$

This is a separable ordinary differential equation.

$$\frac{\hat{f}'(k)}{\hat{f}(k)} = -2\pi k \;\Rightarrow \log\left(\hat{f}(k)\right) = -\pi\, k^2 + \tilde{c} \;\;\Rightarrow \hat{f}(k) = ce^{-\pi k^2}$$

where *c* is a constant.

From homework, we saw $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$.

$$\Rightarrow \hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-i2\pi 0x} dx = \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1 \;\Rightarrow c = 1.$$

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Symmetry Properties of Fourier Transform

If f is real and even then F is real and even.

If f is real and odd then F is imaginary and odd.

If f is real then F is Hermitian (its real part is even and its imaginary part is odd).

Note: you will prove these in homework.

Derivative Theorem

$$\mathcal{F}\left\{f'(x)\right\}(k) = i2\pi k\,\mathcal{F}\{f\}(k)$$

Proof:

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}\mathcal{F}^{-1}\{F\}(x) = \frac{d}{dx}\int_{-\infty}^{\infty} F(k)e^{i\,2\pi\,kx}dk$$

$$= \int_{-\infty}^{\infty} F(k)\frac{d}{dx}e^{i\,2\pi\,kx}dk \;= \int_{-\infty}^{\infty} F(k)(i2\pi k)e^{i\,2\pi\,kx}dk$$

$$\Rightarrow \mathcal{F}\{f'\} = \mathcal{F}\left\{\mathcal{F}^{-1}\{(i2\pi k)F(k)\}\right\} = (i2\pi k)F(k). \qquad \blacksquare$$

Integral Theorem

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(\tau)d\tau\right\}(k) = \frac{\mathcal{F}\{f\}(k)}{i2\pi k} + \frac{1}{2}\mathcal{F}\{f\}(0)\,\delta(k)$$

Proof:

Left for EE 261.

Step Function

$$\mathcal{F}\{u\}(k) = \frac{1}{2}\left(\delta(k) + \frac{1}{i\pi k}\right)$$

Proof:

$$u(t) = \int_{-\infty}^{t} \delta(x)dx$$

Applying the integral property completes the proof. ■

Shift Theorem

$$\mathcal{F}\{f(x-\Delta)\} = e^{-i2\pi k\Delta}\mathcal{F}\{f\}(k)$$

Proof:

$$\mathcal{F}\left\{f(x-\Delta)\right\}(k) = \int_{-\infty}^{\infty} f(x-\Delta)e^{-i2\pi kx}dx.$$ Let $u = x - \Delta$.

$$\mathcal{F}\left\{f(x-\Delta)\right\}(k) = \int_{-\infty}^{\infty} f(u)e^{-i2\pi k(u+\Delta)}dx$$

$$= e^{-i2\pi k\Delta}\int_{-\infty}^{\infty} f(u)e^{-i2\pi ku}dx.$$ ■

Shifted Delta Function

$$\mathcal{F}\{\delta(x-x_0)\}(k) = e^{-i\,2\pi\,kx_0}$$

Proof:

Recall that $\mathcal{F}\{\delta\} = 1$.

Apply the Shift Theorem. ■

Duality

$$\mathcal{F}\left\{\mathcal{F}\left\{f\right\}\right\} = f^{-}$$

Proof:

It suffices to show $\mathcal{F}\{f\} = \mathcal{F}^{-1}\left\{f^{-}\right\}$.

$$\mathcal{F}\{f\}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-i\,2\pi\,kx}\, dx$$

$$\mathcal{F}^{-1}\left\{f^{-}\right\}(k) = \int_{-\infty}^{\infty} f(-s)\, e^{i\,2\pi\,sk}\, ds = \int_{-\infty}^{\infty} f(u)\, e^{-i\,2\pi\,uk}\, du$$

$\blacksquare$

Applications of Duality

We've shown

$$\mathcal{F}\{\delta\} = 1 \qquad \mathcal{F}\{\Pi\} = \text{sinc} \qquad \mathcal{F}\{\delta(x - x_0)\} = e^{-i2\pi k x_0}$$

By duality,

$$\mathcal{F}\{1\} = \delta \qquad \mathcal{F}\{\text{sinc}\} = \Pi \qquad \mathcal{F}\{e^{i2\pi k x_0}\} = \delta(x - x_0)$$

Cosine and Sine

$$\mathcal{F}\{\cos(2\pi x)\}(k) = \frac{1}{2}\left[\delta(k+1) + \delta(k-1)\right]$$

$$\mathcal{F}\{\sin(2\pi x)\}(k) = \frac{i}{2}\left[\delta(k+1) - \delta(k-1)\right]$$

Proof:

$$\mathcal{F}\{\cos(2\pi x)\}(k) = \int_{-\infty}^{\infty} \cos(2\pi x) e^{-i\,2\pi\,kx}\,dx \;= \int_{-\infty}^{\infty} \frac{e^{i2\pi x} + e^{-i2\pi x}}{2}\, e^{-i\,2\pi\,kx} dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-i2\pi(k+1)x} dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{-i2\pi(k-1)x} dx \;= \frac{1}{2}\left[\delta(k-1) + \delta(k+1)\right]$$

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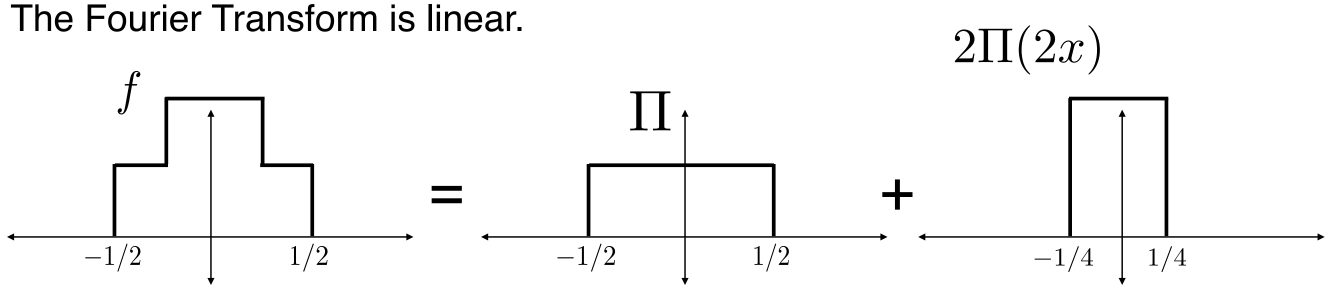
Linearity of the Fourier Transform

The Fourier Transform accepts a function as input and outputs a function.
It's a system!

The Fourier Transform is linear.

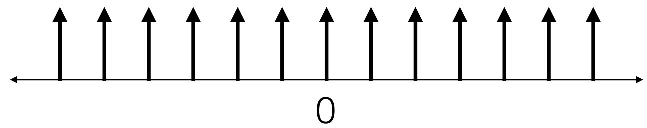


$$\mathcal{F}\{f\}(k) = \mathcal{F}\left\{\Pi(x) + 2\Pi(2x)\right\}(k)$$

$$= \mathcal{F}\left\{\Pi(x)\right\}(k) + \mathcal{F}\left\{2\Pi(2x)\right\}(k) = \mathrm{sinc}(k) + \mathrm{sinc}(k/2).$$

Comb or Shah Function

$$\text{Ш}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$



$$\mathcal{F}\{\text{Ш}\} = \text{Ш}$$

Proof:

Ш is a periodic function with period 1. Therefore, it can be represented by a Fourier series.

$$\text{Ш}(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(i\,2\pi\, nx\right). \qquad c_n = \int_{-1/2}^{1/2} \text{Ш}(x) \exp\left(-i\,2\pi\, nx\right) dx = 1.$$

$$\Rightarrow \text{Ш}(x) = \sum_{n=-\infty}^{\infty} \exp\left(i\,2\pi\, nx\right).$$

Taking the Fourier Transform of both sides:

$$\mathcal{F}\{\text{Ш}\} = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \exp\left(i\,2\pi\, nx\right)\right\} = \sum_{n=-\infty}^{\infty} \mathcal{F}\left\{\exp\left(i\,2\pi\, nx\right)\right\}$$

$$= \sum_{n=-\infty}^{\infty} \delta(k-n) = \text{Ш}.$$

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Scaling Theorem

$$\mathcal{F}\left\{f(\alpha x)\right\} = \frac{1}{|\alpha|}\mathcal{F}\left\{f\right\}\left(\frac{k}{\alpha}\right)$$

Proof:

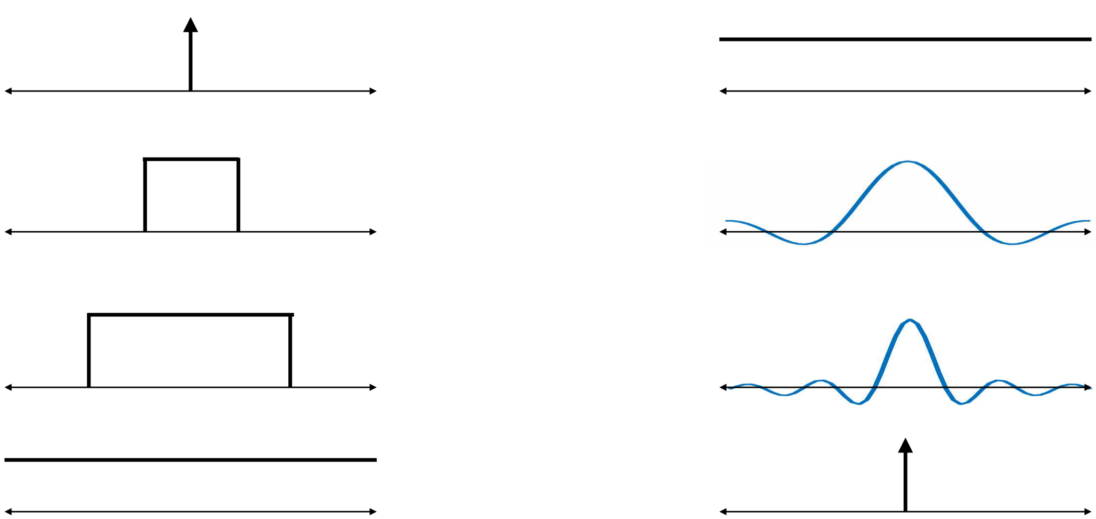
Case 1: $\alpha > 0$

$$\mathcal{F}\left\{f(\alpha x)\right\}(k) = \int_{-\infty}^{\infty} f(\alpha x) e^{-i\,2\pi\,kx} dx = \frac{1}{\alpha}\int_{-\infty}^{\infty} f(u) e^{-i\,2\pi\frac{ku}{\alpha}} du$$

$$= \frac{1}{\alpha}\mathcal{F}\left\{f\left(\frac{k}{\alpha}\right)\right\}(k)$$

When we consider the case where $\alpha < 0$, we complete the proof. ■

The function can be thin before or after the Fourier Transform, but not both.



Heisenberg Uncertainty Principal

The dispersion of a function f about a point a is

$$\Delta_a f = \frac{\int_{-\infty}^{\infty}(x-a)^2|f(x)|^2\,dx}{\int_{-\infty}^{\infty}|f(x)|^2\,dx}.$$

It is a measure of much f is concentrated near *a*.

The HUP states that $\Delta_a f\,\Delta_b F \geq \frac{1}{4}$ for all a, b.

The point: either f or $\mathcal{F}\{f\}$ are spread out.

Complex Conjugation Theorem

$$\mathcal{F}\left\{\bar{f}\right\} = \overline{\mathcal{F}\{f^-\}}$$

Proof:

$$\mathcal{F}\left\{\bar{f}\right\}(k) = \int_{-\infty}^{\infty}\bar{f}(x)\,e^{-i\,2\pi\,kx}dx = \overline{\int_{-\infty}^{\infty}f(x)\,e^{i\,2\pi\,kx}\,dx}$$

$$= \overline{\int_{-\infty}^{\infty}f(-s)\,e^{-i\,2\pi\,ks}\,ds} = \overline{\mathcal{F}\left\{f^-\right\}}(k)$$

$\blacksquare$

Parseval's (Rayleigh's) Theorem

$$E_f = E_F$$

Proof:

Recall: the energy of a function *f* is

$$E_f = \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} f(x)\overline{f(x)} dx = \int_{-\infty}^{\infty} f(x)\, \overline{\mathcal{F}^{-1}\{F(k)\}}\, dx$$

$$= \int_{-\infty}^{\infty} f(x)\, \overline{\int_{-\infty}^{\infty} F(k) e^{i2\pi kx} dk}\, dx = \int_{-\infty}^{\infty} \overline{F(k)} \int_{-\infty}^{\infty} f(x) e^{-i2\pi kx} dx\, dk$$

$$= \int_{-\infty}^{\infty} \overline{F(k)} F(k)\, dk = \int_{-\infty}^{\infty} |F(k)|^2\, dk.$$

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Convolution Theorem

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\}\,\mathcal{F}\{g\}$$

Proof:

$$\mathcal{F}\{f * g\} = \int_{-\infty}^{\infty} (f * g)(x)\, e^{-i2\pi kx} dx = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\gamma) g(x-\gamma) d\gamma \right) e^{-i2\pi kx} dx$$

$$= \int_{-\infty}^{\infty} f(\gamma) \left(\int_{-\infty}^{\infty} g(x-\gamma) e^{-i2\pi kx} dx \right) d\gamma = \int_{-\infty}^{\infty} f(\gamma) \left(e^{-i2\pi k\gamma} \right) G(k) d\gamma$$

$$= G(k) \int_{-\infty}^{\infty} f(\gamma) \left(e^{-i2\pi k\gamma} \right) d\gamma = F(k)\, G(k)$$

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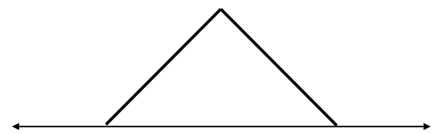
The Fourier Transform Converts Convolution into Multiplication!!!

Tri Function

$$\Lambda(x) = \begin{cases} 1 & \text{if } |x| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$



$$\mathcal{F}\{\Lambda\} = \mathrm{sinc}^2$$

Proof:

$$\Lambda = \Pi * \Pi$$

By applying the convolution theorem, we complete the proof. ■

Generalized Parseval's Theorem

$$\int_{-\infty}^{\infty} f(x)\,\overline{g}(x)\,dx = \int_{-\infty}^{\infty} F(k)\,\overline{G}(k)\,dk$$

Proof:

Left for homework.

Where Are We?

We've seen that a LSI system can be completely characterized by its impulse response.

> The output is equal to the input convolved with the system's impulse response.

We've seen that we can convert the difficult convolution operation into multiplication, which is very easy, with the Fourier Transform.

We now have a pile of theorems to analyze our Systems with in the Fourier domain.