Signal Processing and Linear Systems1

Lecture 7: Analyzing Systems

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Ex: Toy System

Input: $x(t) = 4\cos(4\pi t) + 6\cos(3\pi t) + 2\cos(2\pi t).$

Applied to a system with impulse response $h(t) = 4\,\mathrm{sinc}^2(2t).$

Goal: find the output signal.

Approach:

Determine the Fourier Transform of the input: $X(f).$

Determine the transfer function: $H(f).$

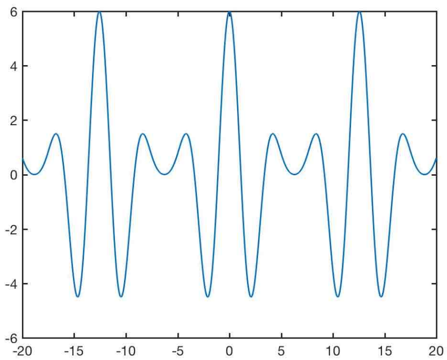
The product is the Fourier Transform of the output: $Y(f).$

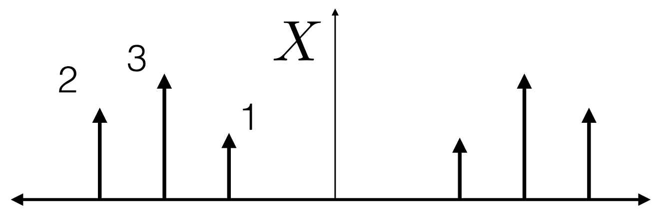
Inverse Fourier Transform to find the output: $y(t) = \mathcal{F}^{-1}(Y)(t).$

$$X(f) = \mathcal{F}\left\{4\cos(4\pi t) + 6\cos(3\pi t) + 2\cos(2\pi t)\right\}(f)$$

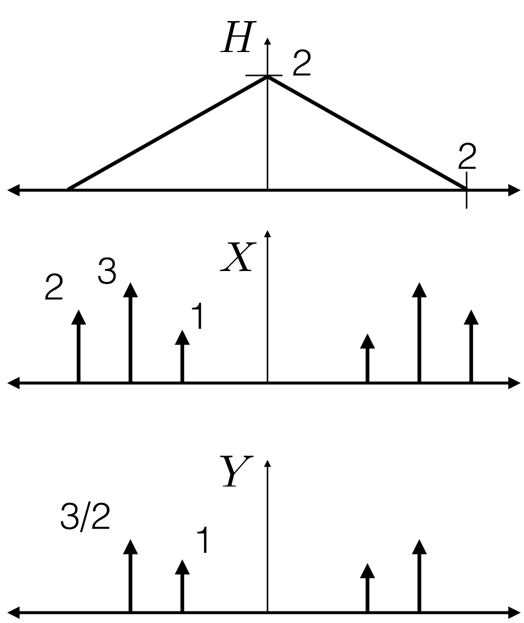
$$= 4\mathcal{F}\left\{\cos(4\pi t)\right\} + 6\mathcal{F}\left\{\cos(3\pi t)\right\} + 2\mathcal{F}\left\{\cos(2\pi t)\right\}(f)$$

$$= 2\left[\delta(f-2) + \delta(f+2)\right] + 3\left[\delta(f-3/2) + \delta(f+3/2)\right] + \left[\delta(f-1) + \delta(f+1)\right]$$





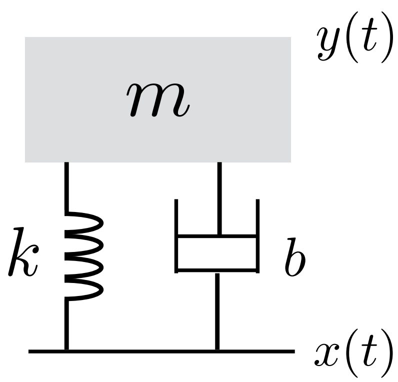
$$H(f) = \mathcal{F}\left\{4\,\mathrm{sinc}^2(2t)\right\}(f) = 2\Lambda(f/2) \qquad Y(f) = H(f)\,X(f)$$



$$Y(f) = (3/2)\left[\delta(f-3/2) + \delta(f+3/2)\right] + \left[\delta(f-1) + \delta(f+1)\right]$$

$$\Rightarrow y(t) = 3\cos(3\pi t) + 2\cos(2\pi t).$$

Ex: Mass on Base



$x(t)$ is the height of the base.

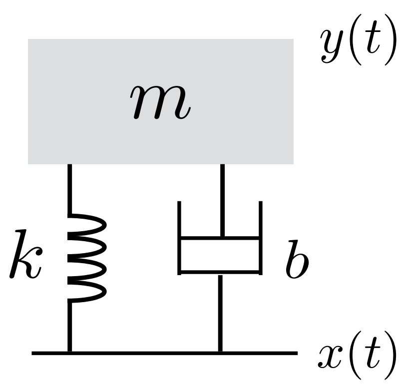
$y(t)$ is the height of the mass.

The mass is attached to the base through a spring and a damper.

This is a simple model for a suspension system for a car.

If *x* is the input and *y* is the output, is this a LSI system?

If so, what is the transfer function of this system?



The force due to the spring is: $F_s = k(x - y)$.

The force due to the damper is: $F_b = b(x - y)'$.

The force due to the mass' acceleration is: $F_a = my''$.

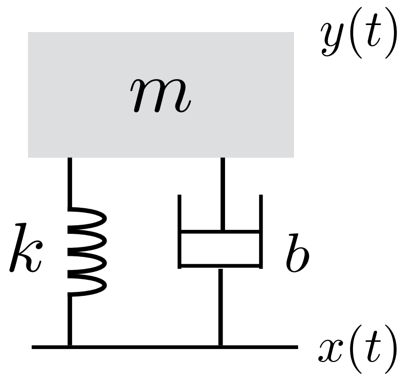
By Newton's 2nd Law: $F_a = F_s + F_b$.

$$\Rightarrow my'' = k(x - y) + b(x - y)'.$$

Rearranging terms: $my'' + by' + ky = bx' + kx$.

Taking the Fourier Transform of both sides:

$$m(i2\pi f)^2 Y(f) + b(i2\pi f)Y(f) + kY(f) = b(i2\pi f)X(f) + kX(f).$$

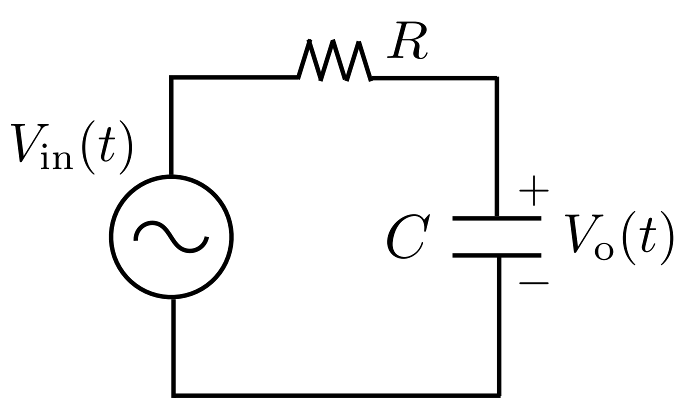


$$m(i2\pi f)^2 Y(f) + b(i2\pi f)Y(f) + kY(f) = b(i2\pi f)X(f) + kX(f).$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{k + i2b\pi f}{-m4\pi^2 f^2 + k + ib2\pi f}.$$

What does this system do (for very low frequencies, very high frequencies)?

Ex: RC Circuit



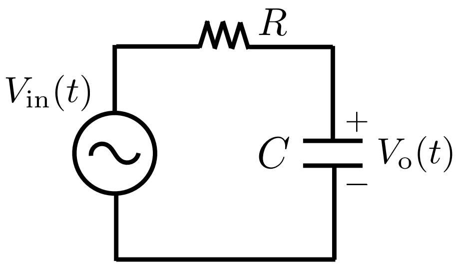
Goal: find the transfer function for this circuit.

In homework, we have seen that the frequency representation of a capacitor is $1/(i2\pi fC)$.

Combining impedances in series yields a total impedance of $R + 1/(i2\pi fC)$.

By distributing voltages across the impedances we get

$$\mathcal{F}\{V_o\} = \mathcal{F}\{V_{in}\}\left(\frac{1/(i2\pi fC)}{R + 1/(i2\pi fC)}\right) \qquad \Rightarrow H(f) = \frac{1/(i2\pi fC)}{R + 1/(i2\pi fC)}.$$

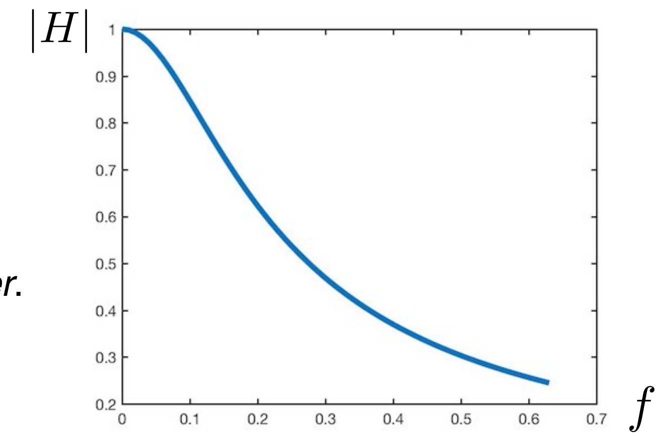


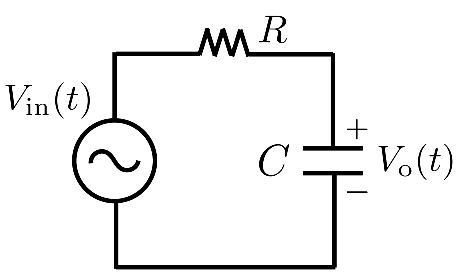
$$\Rightarrow H(f) = \frac{1/(i2\pi fC)}{R + 1/(i2\pi fC)} = \frac{1}{i2\pi fRC + 1}.$$

$$|H(f)| = \frac{1}{\sqrt{1 + (2\pi fRC)^2}}$$

What does this circuit do to very high frequencies?
Very low frequencies?

This is an example of a *Low-Pass Filter*.





What is the output of this circuit to the following input?

$$\mathcal{F}\{V_o\} = H(f)\,\mathcal{F}\{V_{\text{in}}\}(f) = \left(\frac{1}{1 + i2\pi fRC}\right)\mathcal{F}\{V_{\text{in}}\}(f)$$

$$\mathcal{F}\{V_{\text{in}}\}(f) = \mathcal{F}\{\Pi(t - 0.5)\}(f) \;= e^{-i2\pi 0.5 f}\text{sinc}(f)$$

$$\Rightarrow V_{\text{in}}(t) = \mathcal{F}^{-1}\left\{\frac{e^{-i\pi f}\text{sinc}(f)}{1 + i2\pi fRC}\right\}(t)$$

We can split the sine in sinc and use Partial Fraction Expansion to proceed.

Phase Distortion

So far we have been discussing how the magnitude of the input signal is affected.

The transfer function (i.e. the frequency response) can can affect both the magnitude and the phase.

We will now discuss how the phase is altered, and the consequences.

Distortion Free

A causal system with a negative linear phase response is considered distortion free (or distortion-less):

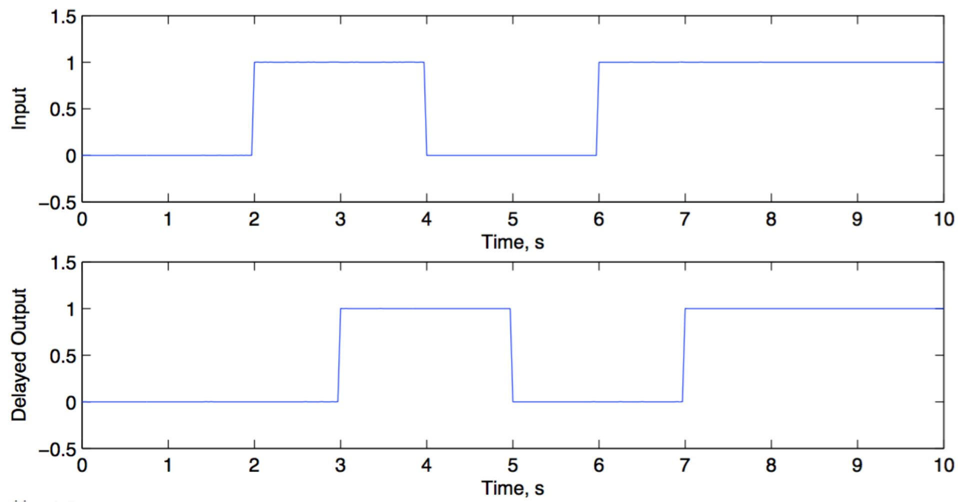
$$\angle H(f) = -2\pi f\, t_d.$$

The consequence of this phase response is to simply delay the output.

By the Fourier Shift Theorem, the delay in time is

$$t_d = \frac{-1}{2\pi} \frac{d}{df} \angle H(f).$$

Linear Phase (Constant Magnitude)



Since the signal arrived out of our system with just a delay, we describe the signal as *distortion free*.

Distortion

If the phase is not linear, then the time delay is no longer constant; it is now a function of frequency:

$$t_d(f) = \frac{-1}{2\pi}\frac{d}{df}\angle H(f).$$

Each frequency gets delayed by a different amount.

The frequency dependent delay is called *group delay*.

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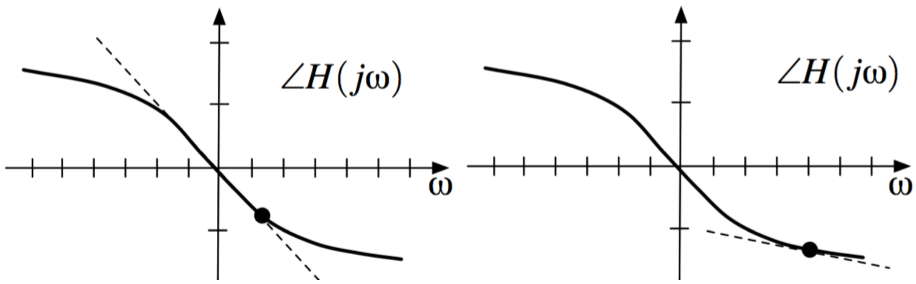

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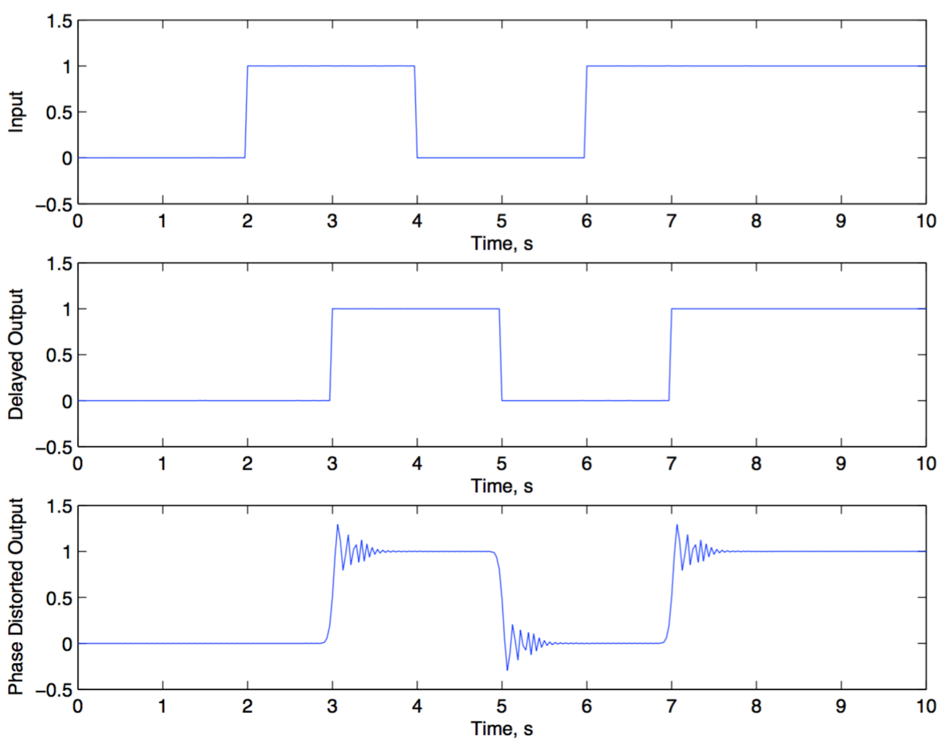

Summary

With the tools we've developed:

We can predict the output for a given input (both the magnitude and the phase).

In homework you will show that we can find the input needed to generate a specific output.

We can determine if a system is stable.