Signal Processing and Linear Systems1

Lecture 9: Properties of the DFT

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Outline

Background

Theorems of the DFT

L2 Norm

$$\|a\|_2 = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Kronecker Delta Vector

$$\delta = (1, 0, 0, \ldots, 0)$$

Matrix Transpose

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & & a_{MN} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{M1} \\ a_{12} & a_{22} & & a_{M2} \\ \vdots & & \ddots & \vdots \\ a_{1N} & a_{2N} & & a_{MN} \end{bmatrix}$$

Matrix Conjugate-Transpose

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & & a_{MN} \end{bmatrix}^* = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{21} & \cdots & \overline{a}_{M1} \\ \overline{a}_{12} & \overline{a}_{22} & & \overline{a}_{M2} \\ \vdots & & \ddots & \vdots \\ \overline{a}_{1N} & \overline{a}_{2N} & & \overline{a}_{MN} \end{bmatrix}$$

Circular Convolution

Suppose $f, g \in \mathbb{C}^N$. Then the circular convolution of *f* and *g* is

$$f \circledast g = \sum_{i=1}^{N} f_i \, \tau_i\{g\}$$

where τ_i is the circular shift operator of i elements.

$$(f \circledast g)[n] = \sum_{m=1}^{N} f[m] \, g\left[(n-m)_{\mathrm{mod}\, N}\right]$$

Recall: for a discrete LSI system S

$$S\{x\} = \sum_{i=1}^{N} x_i \, \tau_i\{h\} = x \circledast h.$$

That is, the output equals the input circularly convolved with the impulse response.

Discrete Fourier Transform

Analysis Equations:

$$\mathrm{DFT}\{\boldsymbol{f}\}[m] = \sum_{n=0}^{N-1} \boldsymbol{f}[n] \, \exp\left(-i \, 2\pi \frac{mn}{N}\right)$$

Synthesis Equations:

$$\mathrm{IDFT}\{\boldsymbol{F}\}[n] = \frac{1}{N} \sum_{m=0}^{N-1} \boldsymbol{F}[m] \, \exp\left(i \, 2\pi \frac{mn}{N}\right)$$

Note: both the DFT and the IDFT can be done by a computer.

Outline

Background

Theorems of the DFT

$$\mathrm{DFT}\{\delta\} = 1$$

Proof:

$$\mathrm{DFT}\{\delta\}[m] = \sum_{n=0}^{N-1} \delta \, \exp\left(-i\, 2\pi \, \frac{mn}{N}\right)$$

$$= \exp\left(-i\, 2\pi \, \frac{0\, m}{N}\right) = 1.$$

The DFT Matrix

$$\mathrm{DFT} : \mathbb{C}^N \to \mathbb{C}^N$$

The Discrete Fourier Transform is a linear transformation from a finite dimensional vector space to a finite dimensional vector space.

Therefore, it can be represented by a matrix: W. $\mathrm{DFT}\{\boldsymbol{f}\} = W\,\boldsymbol{f}$

Similarly, the Inverse Discrete Fourier Transform can be represented by: W^{-1}

$$W^{-1} = \frac{1}{N} W^* \qquad \mathrm{IDFT}\{\boldsymbol{F}\} = W^{-1}\,\boldsymbol{F} = \frac{1}{N} W^*\,\boldsymbol{F}$$

The Unitary DFT Matrix

$$W^{-1} = \frac{1}{N} W^*$$

Let $W_U = \frac{1}{\sqrt{N}} W.$

Then $W_U^{-1} = W_U^*.$

Parseval's Theorem

Claim: if U is a unitary matrix, then $\|U\,x\|_2 = \|x\|_2.$

Proof:

$$\|U\,x\|_2 = \|x\|_2 \quad \Leftrightarrow \quad \|U\,x\|_2^2 = \|x\|_2^2.$$

$$\|U\,x\|_2^2 = (U\,x)^*(Ux) = x^*U^*Ux = x^*Ix = x^*x = \|x\|_2^2.$$

Corollary: $\|\mathrm{DFT}\{x\}\|_2 = \sqrt{N}\,\|x\|_2.$

Convolution Theorem

$$\mathrm{DFT}\{f \circledast g\} = \mathrm{DFT}\{f\} \odot \mathrm{DFT}\{g\}$$

$$\mathrm{DFT}\{f \odot g\} = \mathrm{DFT}\{f\} \circledast \mathrm{DFT}\{g\}$$

Proof: see homework.

For an LSI system, $y = S\{x\}$,

$$Y[m] = X[m]\,H[m]$$

The DFT of the output equals the DFT of the input times the transfer function (the DFT of the impulse response).

Shift Theorem

$$\mathrm{DFT}\{\tau_\Delta\, \boldsymbol{f}\}[m] = \exp\left(-i\, 2\pi \frac{m\,\Delta}{N}\right) \mathrm{DFT}\{\boldsymbol{f}\}[m]$$

Proof:

$$\mathrm{DFT}\{\tau_\Delta\, \boldsymbol{f}\}[m] = \sum_{n=0}^{N-1} \boldsymbol{f}[n-\Delta]\; \exp\left(-i\, 2\pi \frac{mn}{N}\right)$$

$$= \exp\left(-i\, 2\pi \frac{m\,\Delta}{N}\right) \sum_{\tilde{n}=0}^{N-1} \boldsymbol{f}[\tilde{n}]\; \exp\left(-i\, 2\pi \frac{m\,\tilde{n}}{N}\right)$$

$$= \exp\left(-i\, 2\pi \frac{m\,\Delta}{N}\right) \mathrm{DFT}\{\boldsymbol{f}\}[m].$$

Periodicity of the DFT

So far, we've been considering the DFT as a function that maps $\mathbb{C}^N$ to $\mathbb{C}^N$.

But there's nothing stopping us from computing the value of the DFT for frequencies outside of m in {0, …, N-1}.

When we do that (when we think of the domain of the DFT as all integers), we see that the DFT is periodic with fundamental period N.

So we can use any N consecutive values to reconstruct the input vector.

Symmetry of the DFT

DFT{ real and even } = real and even

DFT{ real and odd } = imaginary and odd

DFT{ real } = Hermitian

Note: even and odd are defined according to the periodic extensions of the vectors.