Signal Processing and Linear Systems1

Lecture 11: Multi-Dimensional Fourier Transform

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Fourier Transform

Forward Transform (Analysis Equations):

$$F(k) = \mathcal{F}\{f\}(k) = \int_{-\infty}^{\infty} f(x) \exp\left(-i\,2\pi\,kx\right) dx$$

Inverse Transform (Synthesis Equations):

$$f(x) = \mathcal{F}^{-1}\{F\}(x) = \int_{-\infty}^{\infty} F(k) \exp\left(i\,2\pi\,kx\right) dk$$

Fourier Transform

The Fourier Transform is a system that accepts functions as input and outputs functions.

Input functions: $f : \mathbb{R} \to \mathbb{C}$

Output functions: $F : \mathbb{R} \to \mathbb{C}$

$$F = \mathcal{F}\{f\}$$

F is called the spectrum of f.

Multi-Dimensional Fourier Transform

The Multi-dimensional Fourier Transform is a system that accepts functions as input and outputs functions.

Input functions: $f : \mathbb{R}^N \to \mathbb{C}$

Output functions: $F : \mathbb{R}^N \to \mathbb{C}$

$$F = \mathcal{F}\{f\}$$

F is called the spectrum of f.

Two-dimensional Fourier Transform

Forward Transform (Analysis Equations):

$$F(k_x, k_y) = \mathcal{F}_{2D}\{f\}(k_x, k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i\,2\pi\,(k_x x + k_y y)}\, dx\, dy$$

Inverse Transform (Synthesis Equations):

$$f(x, y) = \mathcal{F}_{2D}^{-1}\{F\}(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_x, k_y)\, e^{i\,2\pi\,(k_x x + k_y y)}\, dk_x\, dk_y$$

Theorem: Fourier Composition

$$F = \mathcal{F}_{2D}\{f\} = \mathcal{F}_{1D,x}\left\{\mathcal{F}_{1D,y}\{f\}\right\}$$

Proof:

$$\mathcal{F}_{1D,x}\left\{\mathcal{F}_{1D,y}\{f\}\right\} = \mathcal{F}_{1D,x}\left\{\int_{-\infty}^{\infty} f(x,y) e^{-i\,2\pi(k_x x + k_y y)} dy\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i\,2\pi\,(k_x x + k_y y)}\, dx\, dy = \mathcal{F}_{2D}\{f\}. \qquad \blacksquare$$

Similarly, the inverse two-dimensional Fourier Transform is the compositions of inverse of two one-dimensional Fourier Transforms.

Multi-dimensional Fourier Transform

Forward Transform (Analysis Equations):

$$F(k) = \mathcal{F}_{ND}\{f\}(k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x)\, e^{-i\,2\pi\,(k\cdot x)}\, dx$$

Inverse Transform (Synthesis Equations):

$$f(x) = \mathcal{F}_{ND}^{-1}\{F\}(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(k)\, e^{i\,2\pi\,(k\cdot x)}\, dk$$

Note: when the context makes the number of dimensions unambiguous, I will often use the $\mathcal{F}$ symbol to mean the appropriate N dimensional Fourier Transform.

Discrete Fourier Transform

Analysis Equations:

$$\mathrm{DFT}\{f\}[m] = \sum_{n=0}^{N-1} f[n] \exp\left(-i\,2\pi \frac{mn}{N}\right)$$

Synthesis Equations:

$$\mathrm{IDFT}\{f\}[m] = \frac{1}{N} \sum_{m=0}^{N-1} F[m] \exp\left(i\,2\pi \frac{mn}{N}\right)$$

Note: both the DFT and the IDFT can be done by a computer.

Two-Dimensional Discrete Fourier Transform

Analysis Equations:

$$F[u,v] = \mathrm{DFT}_{2D}\{f\}[u,v] = \sum_{m=0}^{N-1} \sum_{n=0}^{M-1} f[m,n] e^{-i\,2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}$$

Synthesis Equations:

$$f[m,n] = \mathrm{IDFT}_{2D}\{F\}[m,n] = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F[u,v] e^{i\,2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}$$

There is a composition theorem for the Discrete Fourier Transform.

Two-Dimensional Discrete Fourier Transform

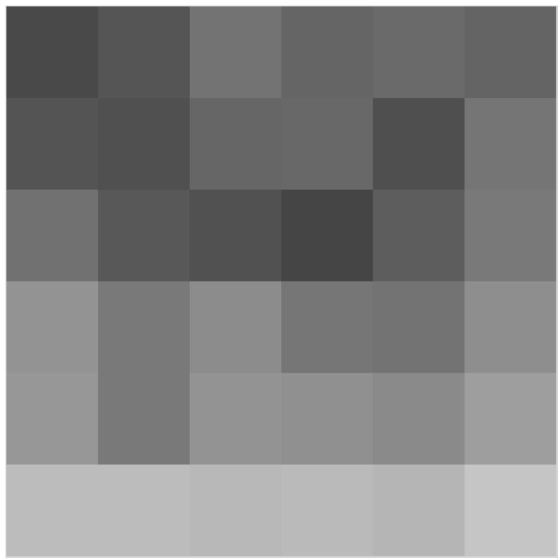
Accepts two-dimensional arrays as inputs and outputs two-dimensional arrays.

Note: another word for a two-dimensional array is *image*.

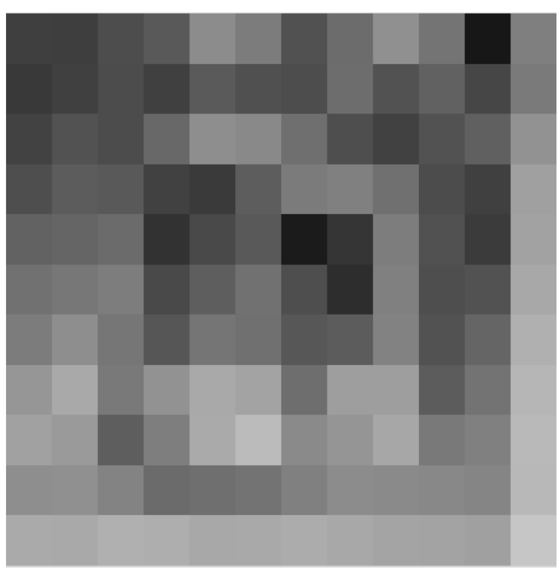
2D Array

57	67	96	82	87	81
66	63	83	85	62	98
94	70	64	53	75	102
129	102	121	99	96	123
133	102	129	125	119	140
174	174	170	172	166	184

Rather than showing the numbers, we can show corresponding colors.
0=black, and 255=white.



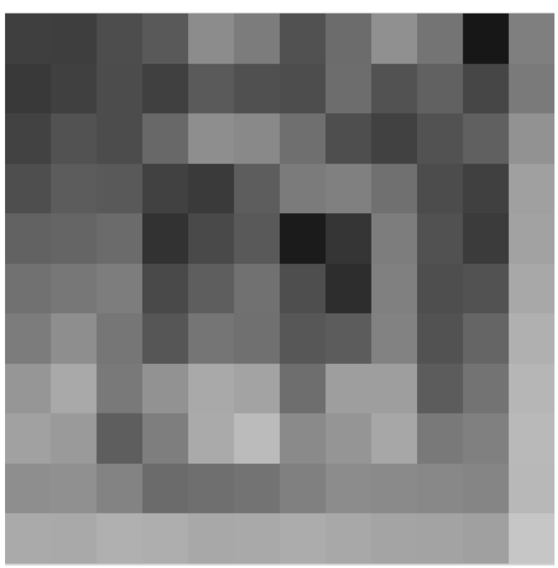
Here's a larger array.
0=black, and 255=white.



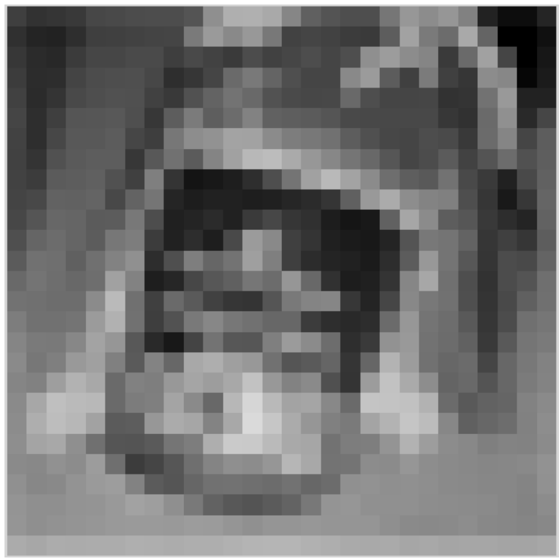
We can always go back to the array of numbers.

48	47	60	71	121	105	64	89	125	97	18	108
43	49	59	49	72	63	60	90	65	78	54	103
51	65	59	85	123	118	92	61	50	65	77	128
61	74	71	50	44	75	104	109	93	59	49	143
79	82	88	38	57	71	21	40	106	64	45	145
94	100	106	57	76	94	61	34	109	61	65	152
105	123	99	68	98	93	69	74	111	65	82	160
132	153	102	128	153	146	91	141	141	74	96	167
144	136	76	107	154	173	119	131	150	102	109	171
123	125	112	88	92	96	109	121	119	117	114	170
154	153	160	158	152	153	156	152	147	146	143	186

Here's a larger array.
0=black, and 255=white.

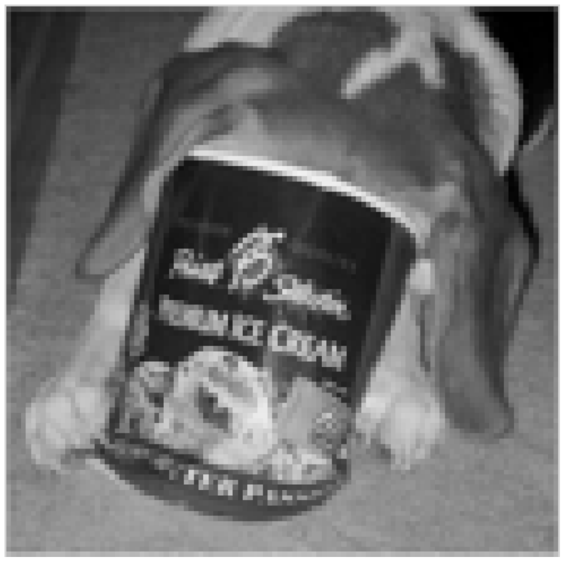


Here's an even larger array. Now we have too many numbers to display on this screen.
0=black, and 255=white.



And larger …
0=black, and 255=white.

And larger
0=black, and 255=white.

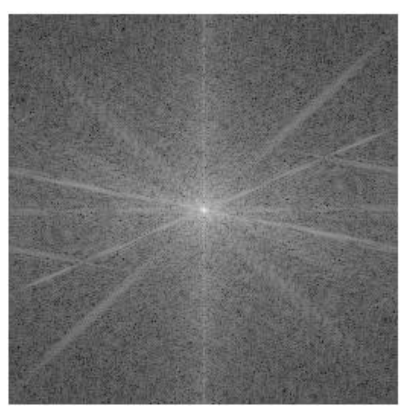


Still larger. At this point, our eye can no longer discern most of the individual pixels.
0=black, and 255=white.

Largest.
0=black, and 255=white.

Two-Dimensional Discrete Fourier Transform

Since an image is just a 2D array, we can compute the two-dimensional Discrete Fourier Transform of an image!



Shown on log scale.