Signal Processing and Linear Systems1

Lecture 12: Two-Dimensional Fourier Transform Theorems

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Two-dimensional Fourier Transform

Forward Transform (Analysis Equations):

$$F(k_x, k_y) = \mathcal{F}_{2D}\{f\}(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, e^{-i\,2\pi\,(k_x x + k_y y)}\, dx\, dy$$

Inverse Transform (Synthesis Equations):

$$f(x, y) = \mathcal{F}_{2D}^{-1}\{F\}(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k_x, k_y)\, e^{i\,2\pi\,(k_x x + k_y y)}\, dk_x\, dk_y$$

2D Delta Function

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)\,dx\,dy = 1$$

$\delta(x,y) = 0$ for all $(x,y) \neq (0,0)$.

Sifting Property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x-x_0, y-y_0)\,dx\,dy = f(x_0,y_0)$$

2D Delta Line Function

Recall: the general equation of a curve in a plane is $\mathcal{C}(x, y) = 0.$

The support of the delta function is a curve.

$$\delta(\mathcal{C}(x, y))$$

Separability of 2D Delta Function

$$\delta(x - x_0, y - y_0) = \delta(x - x_0)\, \delta(y - y_0)$$

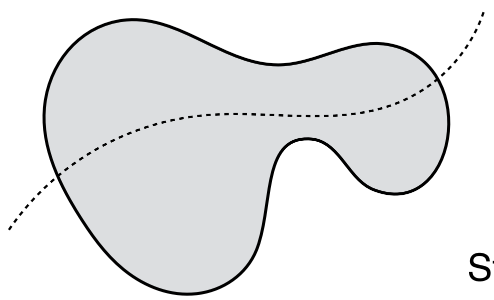
Proof:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, \delta(x - x_0)\, \delta(y - y_0) dx\, dy \; = \int_{-\infty}^{\infty} f(x_0, y)\, \delta(y - y_0)\, dy$$

$$= f(x_0, y_0). \qquad \blacksquare$$

Strength of 2D Delta Line Function

Consider a double integral encompassing an area of the curve:



$$\text{Strength(x,y)} = \frac{\displaystyle\lim_{\text{Area}\to 0} \iint_{\text{Area}} \delta\left(\mathcal{C}(x,y)\right)\, dx\, dy}{\text{Length of curve in integrated area}}$$

Recall: $\delta(ax) = (1/|a|)\,\delta(x)$

The strength of a delta line function is $|a|$.

Consider the following delta line function: $\delta(f(x,y))$.

The strength of the delta line function at the point (x,y) is $|\nabla f(x,y)|$ where ∇ means gradient.

Example

$$\mathcal{F}_{2D}\{\delta(x)\} \; = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x) e^{-i2\pi(k_x\, x + k_y\, y)}\, dx\, dy$$

$$= \int_{-\infty}^{\infty} e^{-i2\pi\, k_y y} dy \; = \delta(y).$$

Separable Functions

A function $f : \mathbb{R}^2 \to \mathbb{C}$ is separable means there exists $f_x, f_y : \mathbb{R} \to \mathbb{C}$ such that $f(x,y) = f_x(x)\, f_y(y)$ for all x, y.

For a separable function, $\mathcal{F}_{2D}\{f\} = \mathcal{F}_{1D}\{f_1\}\, \mathcal{F}_{1D}\{f_2\}$.

Proof:

$$\mathcal{F}_{2D}\{f\}(k_x, k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_x(x)\, f_y(y)\, e^{-i2\pi\, k_x x + k_y y}\, dx\, dy$$

$$= \int_{-\infty}^{\infty} f_x(x) e^{-i2\pi\, k_x x} dx \int_{-\infty}^{\infty} f_y(y)\, e^{-i2\pi\, k_y y}\, dy.$$

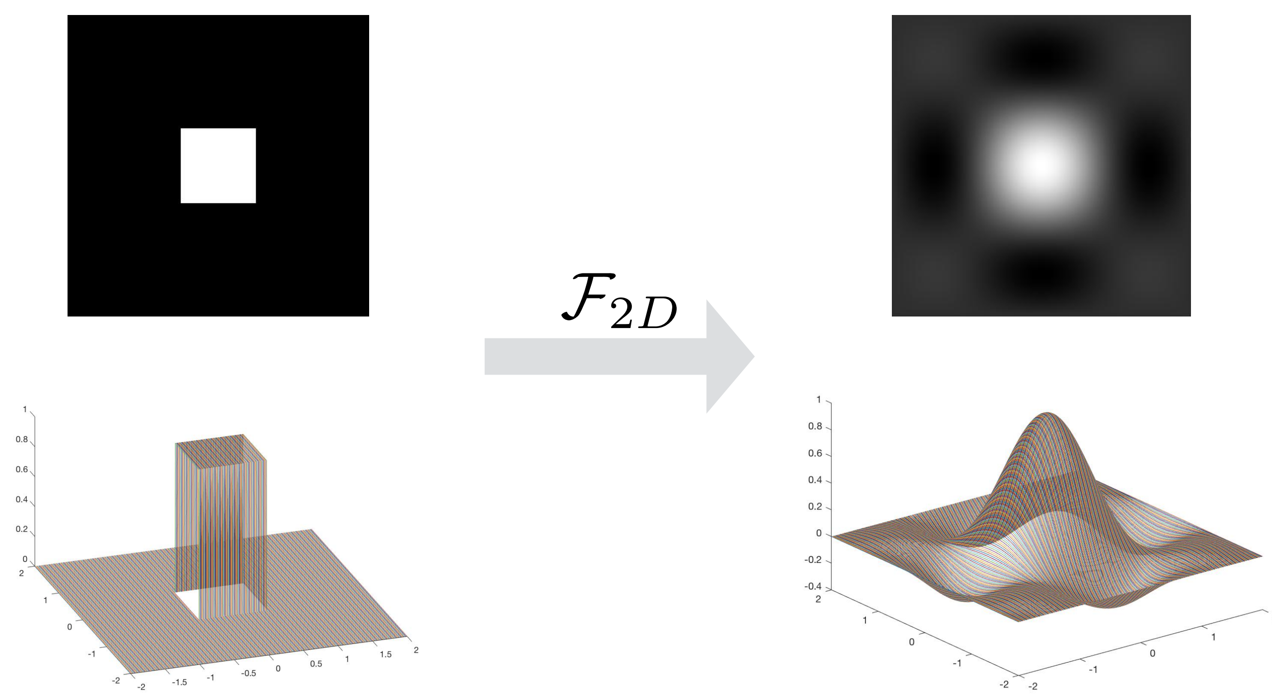
$\blacksquare$

Example

$$\Pi(x,y) = \begin{cases} 1 & \text{if } |x| < 0.5 \text{ and } |y| < 0.5 \\ 0 & \text{otherwise} \end{cases}$$

$$\Pi(x,y) = \Pi(x)\,\Pi(y)$$

$$\begin{aligned} \mathcal{F}_{2D}\{\Pi(x,y)\}(u,v) &= \mathcal{F}_{1D}\{\Pi(x)\}(u)\,\mathcal{F}_{1D}\{\Pi(y)\}(v) \\ &= \operatorname{sinc}(u)\operatorname{sinc}(v). \end{aligned}$$



Theorem: Fourier Composition

$$F = \mathcal{F}_{2D}\{f\} = \mathcal{F}_{1D,x}\left\{\mathcal{F}_{1D,y}\{f\}\right\}$$

Proof:

$$\mathcal{F}_{1D,x}\left\{\mathcal{F}_{1D,y}\{f\}\right\} = \mathcal{F}_{1D,x}\left\{\int_{-\infty}^{\infty} f(x,y)e^{-i\,2\pi(k_y y)}dy\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,e^{-i\,2\pi\,(k_x x + k_y y)}\,dx\,dy = \mathcal{F}_{2D}\{f\}. \qquad \blacksquare$$

Similarly, the inverse two-dimensional Fourier Transform is the compositions of inverse of two one-dimensional Fourier Transforms.

Example

$$f(x,y) = \Pi(x)\,\delta(x-y).$$

$$\begin{aligned}\mathcal{F}_{2D}\{f\} &= \mathcal{F}_{1D,x}\left\{\mathcal{F}_{1D,y}\left\{\Pi(x)\,\delta(y-x)\right\}\right\}\\ &= \mathcal{F}_{1D,x}\left\{\Pi(x)\,\mathcal{F}_{1D,y}\left\{\delta(y-x)\right\}\right\}\\ &= \mathcal{F}_{1D,x}\left\{\Pi(x)\,e^{-i2\pi xv}\right\}\\ &= \mathrm{sinc}(u-v).\end{aligned}$$

Scaling Theorem

$$\mathcal{F}\{f(\alpha x, \beta y)\}(k_x, k_y) = \frac{1}{|\alpha\beta|}\mathcal{F}\{f\}\left(\frac{k_x}{\alpha}, \frac{k_y}{\beta}\right)$$

Proof: Left as an exercise.

More general scaling theorem: let $A \in \mathbb{R}^{2\times 2}$.

$$\mathcal{F}\left\{f\left(A\begin{bmatrix}x\\y\end{bmatrix}\right)\right\}(k_x, k_y) = \frac{1}{|\det(A)|}\mathcal{F}\{f\}\left(A^{-T}\begin{bmatrix}k_x\\k_y\end{bmatrix}\right)$$

Corollary: Rotation Theorem

A rotation is defined as matrix whose determinant is 1.

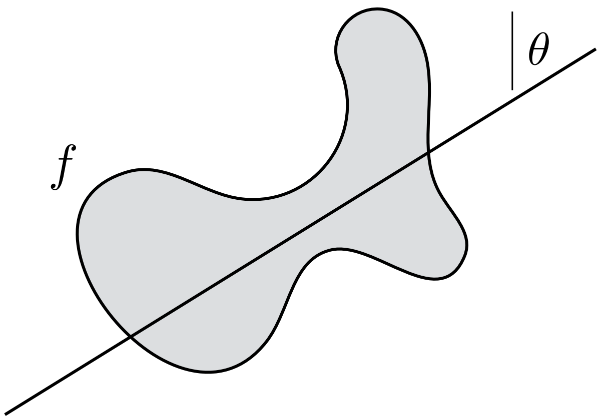
A rotation matrix is orthogonal: its inverse is its transpose.

Let $R \in \mathbb{R}^{2\times 2}$ be a rotation matrix.

$$\mathcal{F}\left\{f\left(R\begin{bmatrix}x\\y\end{bmatrix}\right)\right\}(k_x, k_y) = \mathcal{F}\{f\}\left(R\begin{bmatrix}k_x\\k_y\end{bmatrix}\right)$$

In words: “Rotating the function rotates the Fourier Transform of the function.”

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, \delta\left(x\cos(\theta) + y\sin(\theta) - l\right)\, dx\, dy$$

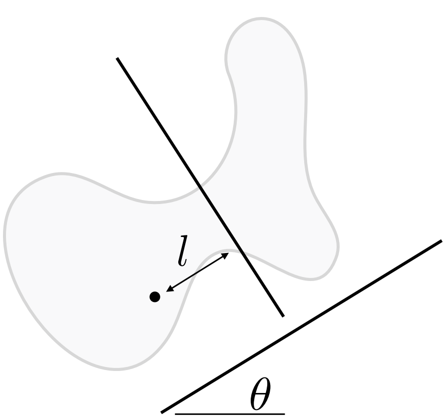


This integrates f along the line defined by the support of the delta.

It's another way of integrating along a curve.

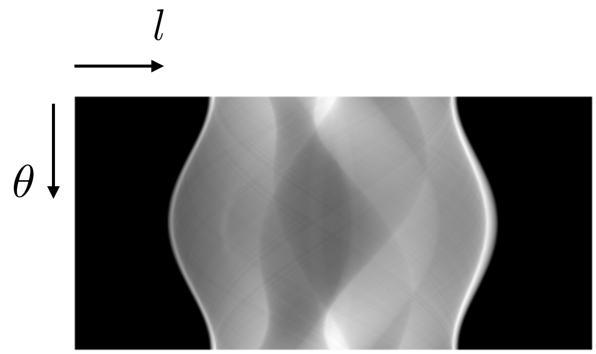
Radon Transform

$$\mathcal{R}\{f\}(l,\theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, \delta\left(x\cos(\theta) + y\sin(\theta) - l\right)\, dx\, dy$$



The value of $\mathcal{R}\{f\}(l,\theta)$ is the integration of the function *f* along a line a distance l away from the origin at an angle θ .

Radon Transform



Sinogram

Consider the value of the Fourier Transform

$$
\begin{aligned}
F(u,v)\big|_{v=0} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i2\pi(ux+0y)}dx\,dy \\
&= \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} f(x,y)dy\right]}_{\text{Projection along vertical lines}} e^{-i2\pi\,ux}dx
\end{aligned}
$$

$$
\mathcal{R}\{f\}(x,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\gamma,y)\delta(x-\gamma)\,dy\,d\gamma = \int_{-\infty}^{\infty} f(x,y)\,dy
$$

$$F(u,v)|_{v=0} = \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} f(x,y)\delta(x)\, dy\right]}_{\text{Projection along vertical lines}} e^{-i2\pi\, ux} dx$$

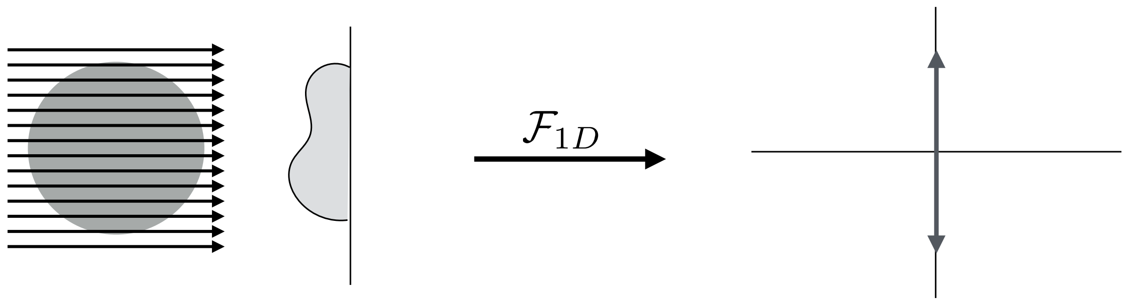
$$F(u,0) = \mathcal{F}_{1D}\left\{ \mathcal{R}\{f\}(l,0)\right\}$$

The horizontal line through the 2D Fourier Transform equals the 1D Fourier Transform of the vertical projection.

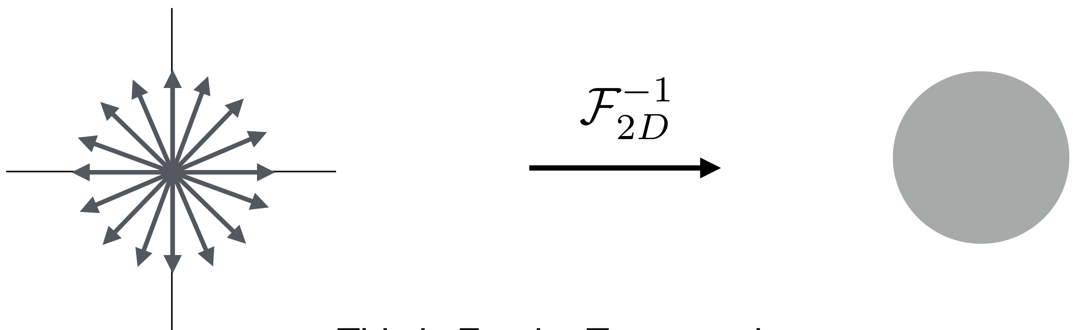
Since rotating the function rotates the Fourier Transform, the same is true for projections at all angles.

Fourier Slice Theorem

The Fourier Transform of a Projection is a Slice of the Fourier Transform.



Taking the Fourier Transform of projections at different angles gives many different lines of the Fourier Transform.



This is Fourier Tomography.