An Introduction to the MRI Signal Equation

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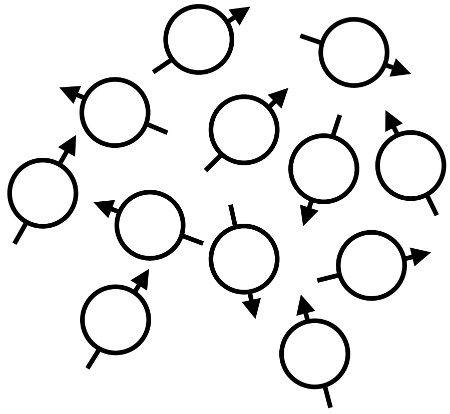
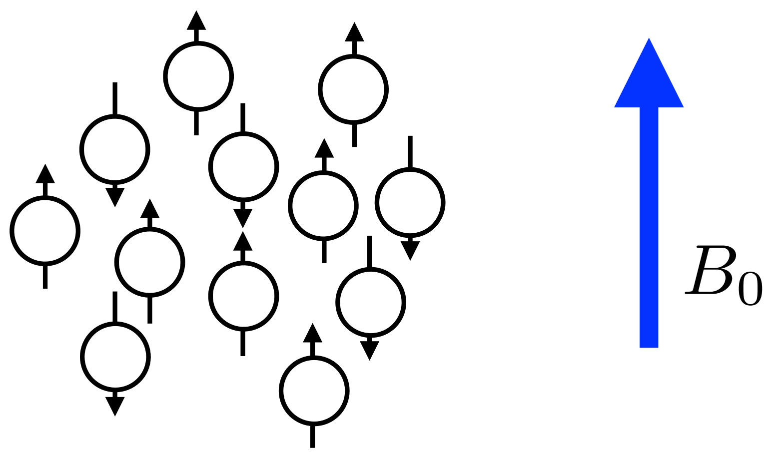
A particle is a bunch of ^1H atoms.

Each atom has a magnetic vector.

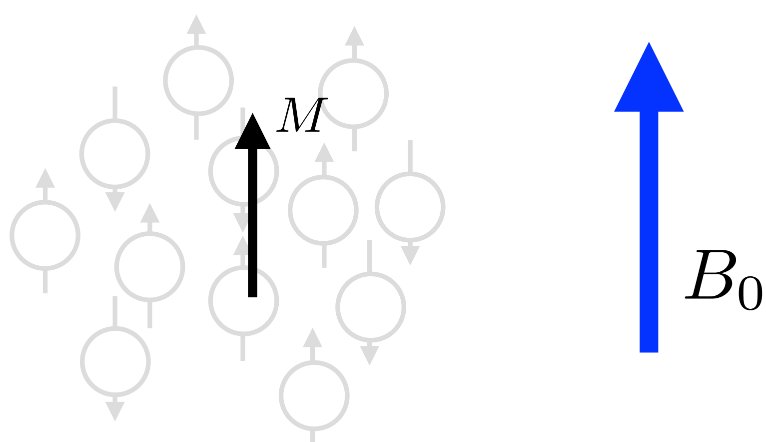
The vectors are randomly distributed.



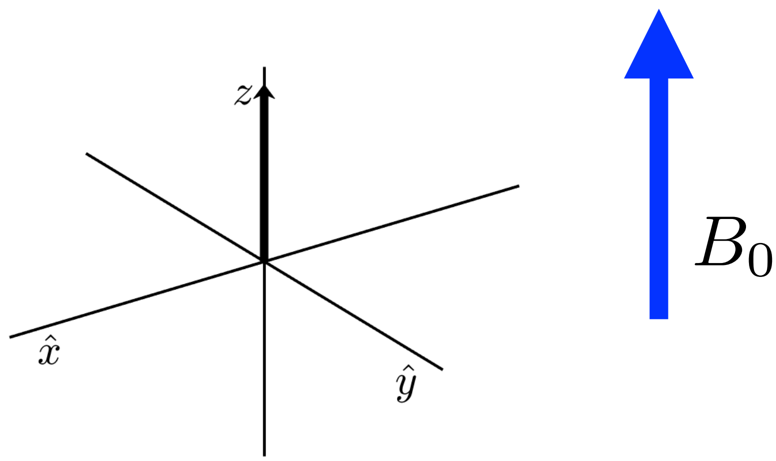
The vectors preferentially align with a large external magnetic field.



This results in a net magnetic vector M in the direction of B_0.



We define a coordinate system centered on this particle.



Bloch Equation

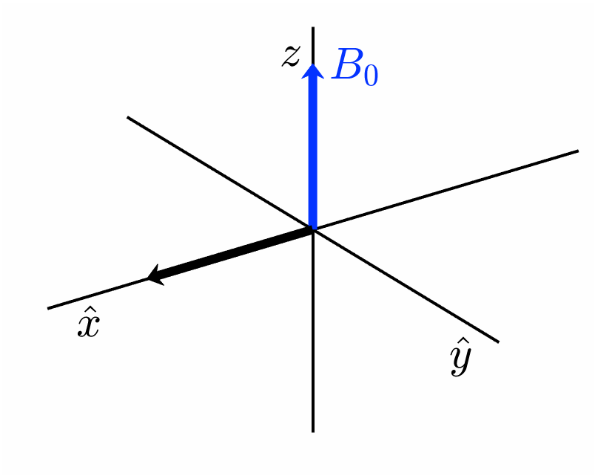
Governs the behavior of a magnetic particle in the presence of an external magnetic field.

$$M'(t) = \gamma\, M(t) \times B(t) - \begin{bmatrix} M_x(t)/T_2 \\ M_y(t)/T_2 \\ (M_z(t) - M_0)/T_1 \end{bmatrix}$$

γ is called the gyromagnetic ratio.

43 MHz / T for Hydrogen

Bloch's Equation dictates a left handed precession.

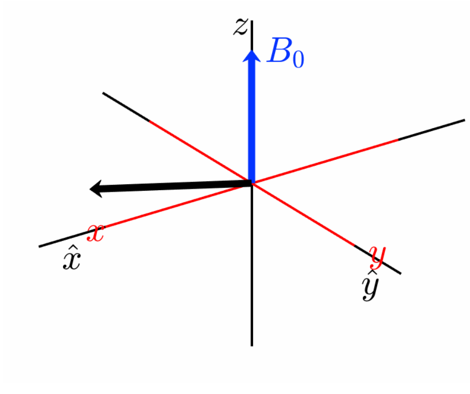


The frequency of precession is called the Larmour frequency

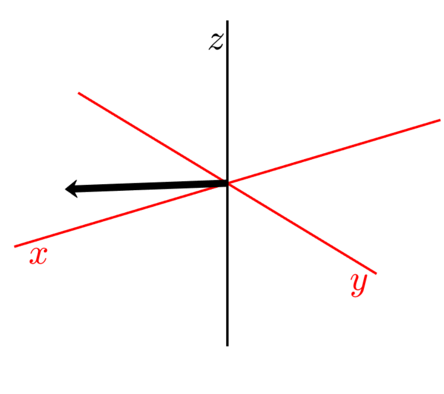
$$\omega = \gamma B_0$$

Consider a separate coordinate system that rotates at the Larmour frequency.

Lab Frame



Rotating Frame



We'll only consider the rotating frame from now on.

Free Induction Decay (FID)

$$M'(t) = \gamma\, M(t) \times B(t) - \begin{bmatrix} M_{x'}(t)/T2 \\ M_{y'}(t)/T2 \\ (M_{z'}(t) - M_0)/T1 \end{bmatrix}$$

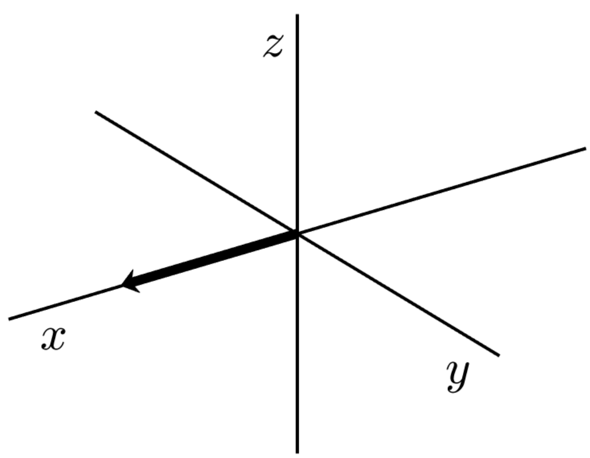
In the rotating frame with $B(t) = (0, 0, B_0)$

$$\boldsymbol{M(t)} = \underbrace{\begin{bmatrix} e^{-t/T2} & & \\ & e^{-t/T2} & \\ & & e^{-t/T1} \end{bmatrix}}_{\text{Relaxation}} M(0) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ M_0\left(1 - e^{-t/T1}\right) \end{bmatrix}}_{\text{Recovery}}$$

Cartoon showing FID of heart tissue.

$T1 \approx 1030\,\mathrm{ms}$

$T2 \approx 40\,\mathrm{ms}$



Stanisz, Greg J., et al. "T1, T2 relaxation and magnetization transfer in tissue at 3T." Magnetic resonance in medicine 54.3 (2005): 507-512.

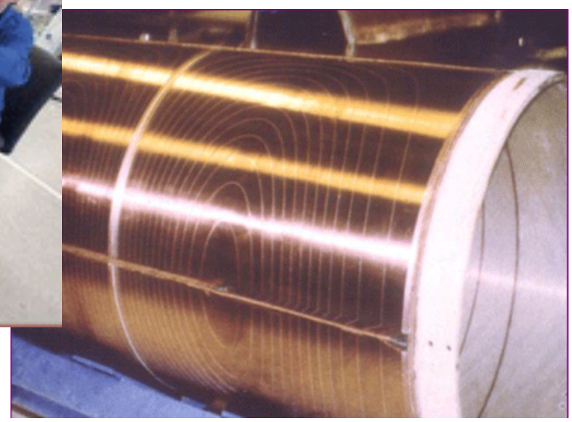
B_0 field constantly on, generated by current passing through cryogen cooled superconducting coils

http://www.qhc.on.ca/mri-p760.php

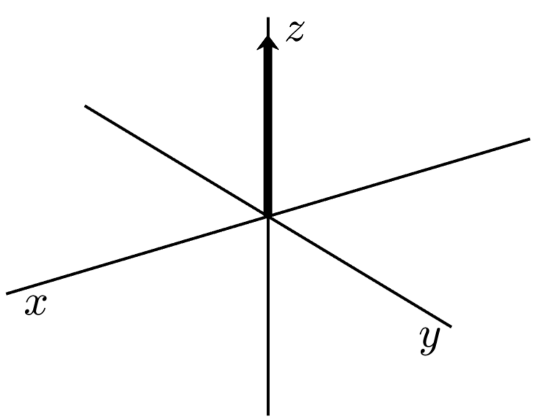
Patient enters the bore; the magnetic state of static tissue reaches equilibrium.

https://www.reference.com/health/mri-scan-like-61ccaabd47ac2d75

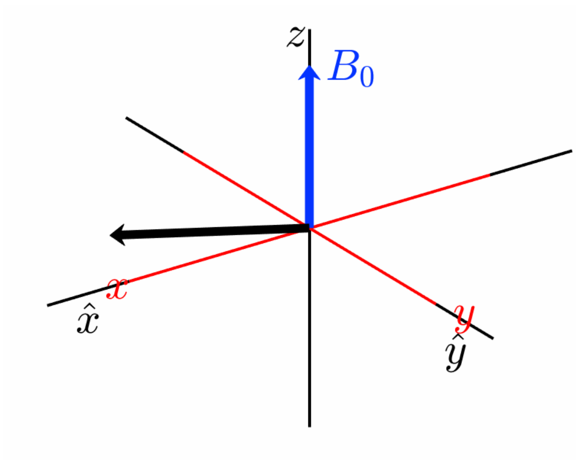
MRI Field Altering Coils



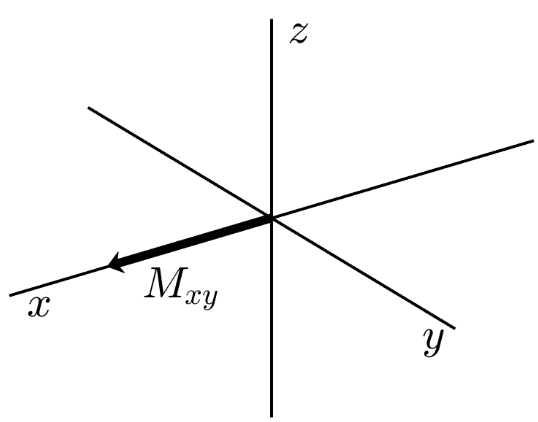
Excitation: an RF pulse generates a magnetic field B_1, which rotates M into the xy plane.



The M vector rotates in the xy-plane of the lab frame. This changing magnetic field induces a voltage in the MRI coils.



The xy component of M is represented as a complex number M_{xy}.



$$M_{xy} = M_x + i\, M_y$$

Patient is in the bore.

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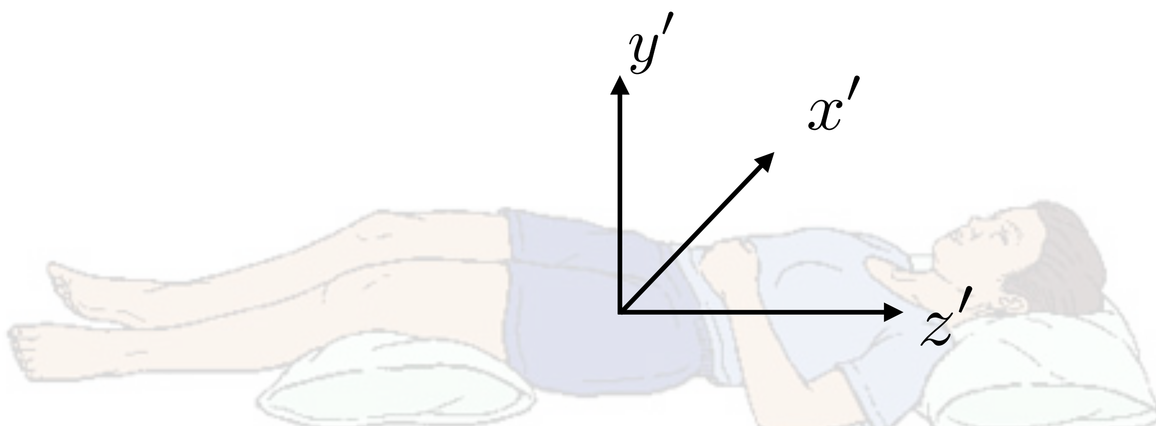

We define a coordinate system centered on the patient.

Gradient coils change z component of magnetic field as a function of space.

$$B_{z'}(r', t) = x'G_{x'}(t) + y'G_{y'}(t) + z'G_{z'}(t)$$

Example: $G_{x'} = G_{y'} = 0$, and $G_{z'} > 0$.

z'

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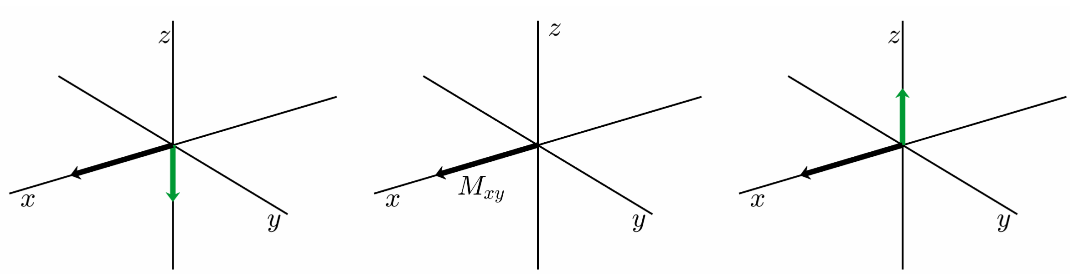

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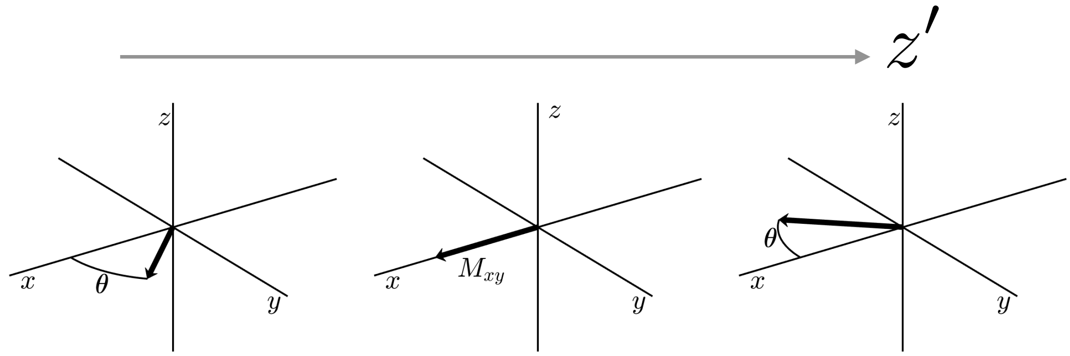

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Precessing magnetization in the xy plane induces a voltage in the receiver coil.

$$\begin{aligned} s(t) &\approx \sum_P M_{xy,j} e^{i\phi_j(t)} \, \Delta v_j \\ &\approx \sum_P M_{xy,j} \exp\left(-i\gamma\left(\int_0^t r'_j(\tau) \cdot G(\tau) d\tau\right)\right) \Delta v_j \end{aligned}$$

Assuming the particles don't move, and taking the limit as the particle size goes to 0,

$$s(t) = \int_{r'} M_{xy}(r') \exp\left(-i\gamma\left(r' \cdot \int_0^t G(\tau) d\tau\right)\right) dr'$$

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$$s(t) = \int_r M_{xy}(r) \exp\left(-i\gamma\left(r \cdot \int_0^t G(\tau) d\tau\right)\right) dr$$

Let $k_{x'}(t) = \frac{\gamma}{2\pi} \int_0^t G_{x'}(\tau)\, d\tau$. Similarly for y', z'.

$$\boxed{s(t) = \int_{r'} M_{xy}(r') \exp\left(-i\, 2\pi\, \gamma r' \cdot k\right) dr'}$$

This is a Fourier transform!

We can recover M_{xy} with an inverse.

With relaxation, the signal equation becomes

$$s(t) = \int_{r'} E(t)\, M_{xy}(r') \exp\left(-i\, 2\pi\, \gamma r' \cdot k\right) dr'$$

The apodization by E , which is a function of T1 and T2, is a source of signal contrast in MRI images.

Spin Warp (or 2DFT)