A vector space V over a field $\mathbb{F}$ is a set where

$$u + (v + w) = (u + v) + w$$
$$u + v = v + u$$
$$\exists\ 0 \in V \text{ s.t. } v + 0 = v \quad \forall\ v \in V$$
$$\forall\ v \in V\ \exists\ (-v) \in V \text{ s.t. } v + (-v) = 0.$$

$$a(bv) = (ab)v$$
$$1v = v$$
$$a(u+v) = au + av$$
$$(a+b)v = av + bv.$$

Examples: $\mathbb{R}^5$, periodic functions with period T, the set of all functions from $\mathbb{R}$ to $\mathbb{C}$.

A set S generates V means $\text{span}(S) = V$.

A basis is a set β with the smallest possible number of vectors that generates V.

The dimension of a vector space is the size of its bases.
(Thm: all bases have the same size.)

Ex: $\dim(\mathbb{R}^5) = 5$
$\dim(\text{polynomials of order } 5) = 6$.

An inner product satisfies:

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle.$$
$$\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$$
$$\langle v, w \rangle = \overline{\langle w, v \rangle}$$
$$\cancel{\langle v, w \rangle \geq 0 \text{ and}}$$
$$\langle v, v \rangle \geq 0 \text{ and } \langle v, v \rangle = 0 \text{ iff } v = 0.$$

Given a vector x and an orthonormal basis β, how do we find the coordinates of x (i.e. how do we find the linear coefficients)?

Answer: Fourier's Trick.

$$\langle x, \beta_i \rangle = \langle a_1 \beta_1 + a_2 \beta_2 + \cdots + a_i \beta_i + \cdots + a_N \beta_N, \beta_i \rangle$$
$$= a_1 \langle \beta_1, \beta_i \rangle + \cdots + a_i \langle \beta_i, \beta_i \rangle + \cdots + a_N \langle \beta_N, \beta_i \rangle$$
$$= a_i.$$

For functions with period T,

$$\beta = \left\{ e^{i 2\pi \frac{nx}{T}} : n \in \mathbb{Z} \right\}.$$

The inner product is: $\langle f, g \rangle = \frac{1}{T} \int_0^T f(t) \overline{g(t)}\, dt$

Let the m^{th} coefficient be F_n.

Then $F_m = \langle f(t), e^{+i 2\pi \frac{mt}{T}} \rangle$

$$= \frac{1}{T} \int_0^T f(t)\, e^{-i 2\pi \frac{mt}{T}}\, dt.$$

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For non-periodic functions,

$$\beta = \left\{ e^{i 2\pi k x} : k \in \mathbb{R} \right\}.$$

The inner product is: $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)}\, dx.$

☐ Explain changing coordinates with matrix inverses.

Consider the circular shift operator: $S : \mathbb{C}^N \to \mathbb{C}^N$.

$$S(x) = [\, x_{N-1} \;\; x_0 \;\; x_1 \cdots x_{N-2} \,].$$

This is a linear transformation from a finite dimensional vector space to a finite dimensional vector space. Thus, it can be represented as matrix multiplication. We will also call this matrix S. It is easy to see that the matrix is

$$S = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & & & \vdots \\ 0 & 1 & \ddots & \\ \vdots & & & \\ 0 & \cdots & 0 \;\; 1 & 0 \end{bmatrix}.$$

An operator T is circularly shift invariant means

$$T(S(x)) = S(T(x)) \quad \forall\, x \in \mathbb{C}^N, \cancel{k \in \mathbb{Z}}.$$

where $T : \mathbb{C}^N \to \mathbb{C}^N$.

Lemma: If T is circularly shift invariant, then $T(S^k(x)) = S^k(T(x)) \quad \forall\, k \in \mathbb{Z}$.

Let $\{e_0, e_1, \cdots, e_{N-1}\}$ be the standard basis. Then $x = x_0 e_0 + x_1 e_1 + \cdots + x_{N-1} e_{N-1} = \sum_{n=0}^{N-1} x_n e_n$.

$$T(x) = T\left(\sum_{n=0}^{N-1} x_n e_n\right) = \sum_{n=0}^{N-1} T(x_n e_n) = \sum_{n=0}^{N-1} x_n T(e_n).$$

Note: $e_m = S^m(e_0)$.

$$\Rightarrow T(x) = \sum_{n=0}^{N-1} x_n T(S^n(e_0)) = \sum_{n=0}^{N-1} x_n S^n(T(e_0))$$

$$= \sum_{n=0}^{N-1} x_n S^n(h). \quad = h \circledast x.$$

This also can be represented with matrix multiplication:

$$T(x) = \begin{bmatrix} h_0 & h_{N-1} & \cdots & h_1 \\ h_1 & h_0 & & h_2 \\ h_2 & h_1 & & \vdots \\ \vdots & \vdots & & \vdots \\ h_{N-1} & h_{N-2} & \cdots & h_0 \end{bmatrix} x.$$

The matrix for T is a circulant matrix.

Another way to write it:

$$T(x) = \sum_{n=0}^{N-1} h_n S^n(x) = \underbrace{\left(\sum_{n=0}^{N-1} h_n S^n\right)}_{\text{This is the circulant matrix.}} x$$

I will also call this matrix T. $T(x) = Tx$, where $T = \sum_{n=0}^{N-1} h_n S^n$.

We will find the eigenvectors of T.

Lemma: A vector v is an eigenvector of T if v is an eigenvector of S.

Proof: $T(v) = \sum_{n=0}^{N-1} h_n S^n(v) = \sum_{n=0}^{N-1} h_n \lambda^n v = \left(\sum_{n=0}^{N-1} h_n \lambda^n\right) v.$

So, if we find an eigenbasis of S, then we have found an eigenbasis of T.

Let v be an eigenvalue of S. Then $Sv = \lambda v$. Since S is invertible, $\lambda \neq 0$.

$\cancel{\Rightarrow x_{N-1} = \lambda x_0,\ x_0 = \lambda}$

$\Rightarrow v_{N-1} = \lambda v_0,\ v_0 = \lambda v_1,\ v_1 = \lambda v_2, \dots, v_{N-2} = \lambda v_{N-1}.$

if $v_0 = 0$, then $v = 0$. $\therefore v_0 \neq 0$.

$\Rightarrow$ Let's scale, then, so that $v_0 = 1$.

$\Rightarrow v_0 = \lambda^N v_0$ or $\lambda^N = 1$.

~~As we~~ There are N unique N^{th} roots of 1.

Let $\omega = \exp(i\,2\pi/m)$. Let $\lambda_j = \omega^{-j}$. This gives us the eigenvector

$$v_j = (1, \lambda^{-1}, \lambda^{-2}, \dots, \lambda^{-(N-1)})$$
$$= (1, \omega^j, (\omega^j)^2, \dots, (\omega^j)^{N-1}).$$

There are N different vetors (N different values of j). This is an orthogonal eigen basis.

These are the DFT vectors.

We can normalize these vectors: $u_j = v_j / \|v_j\|$.

$\|v_j\| = \sqrt{N} \quad \forall j \Rightarrow u_j = v_j / \sqrt{N}$.

The set $\{u_0, u_1, \ldots, u_{N-1}\}$ is an orthonormal eigenbasis. It is called the "Discrete Fourier Basis."

Let F_u be the $N \times N$ matrix whose j^{th} column is u_j. Then F_u is a unitary matrix.
The change of coordinates from the standard basis to the Discrete Fourier basis is $F_u^{-1} = F_u^*$.

$\hat{x} = F_u^* x$. $\hat{x}$ is called the "Unitary Discrete Fourier Transform of x."

We know that $\{u\}$ is a set of orthonormal eigenvectors of T.

$T u_j = \gamma_j u_j \Leftrightarrow T F_u = F_u \Gamma \Leftrightarrow T = F_u \Gamma F_u^*$,

where $\Gamma = diag(\gamma_0, \gamma_1, \cdots, \gamma_{N-1})$.

I.e. T is diagonalized by F_u.

$T x = F_u \Gamma F_u^* x \Leftrightarrow (h \circledast x) = F_u \Gamma F_u^* x$

$\Leftrightarrow F_u^* (h \circledast x) = \Gamma F_u^* x$

$\Leftrightarrow \widehat{h \circledast x} = \Gamma \hat{x}$. This is the convolution theorem.