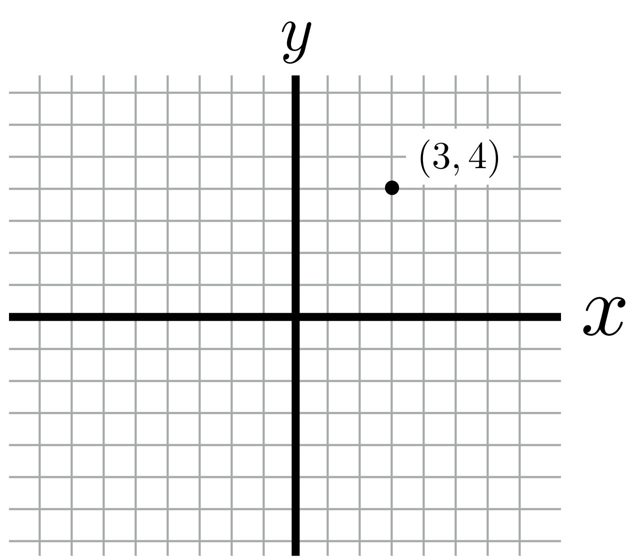
Fundamentals

Math Lecture 1

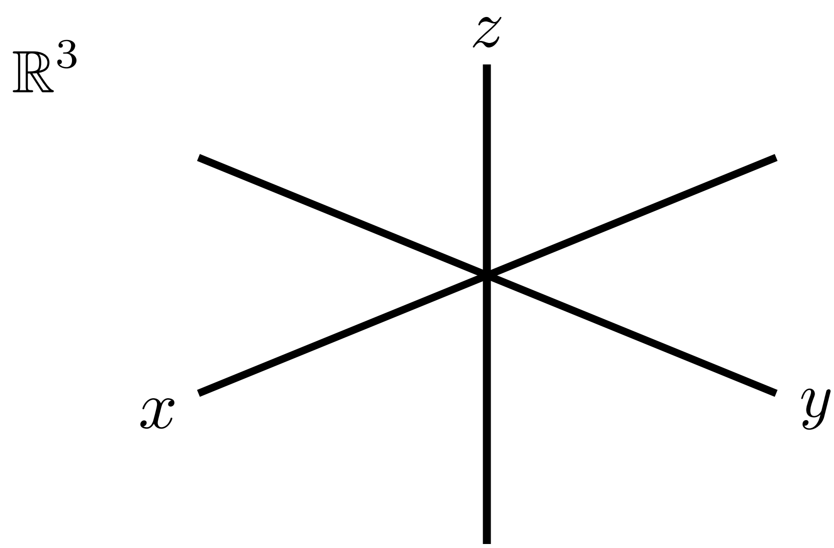
Nicholas Dwork

The Euclidean Plane

$\mathbb{R}^2$

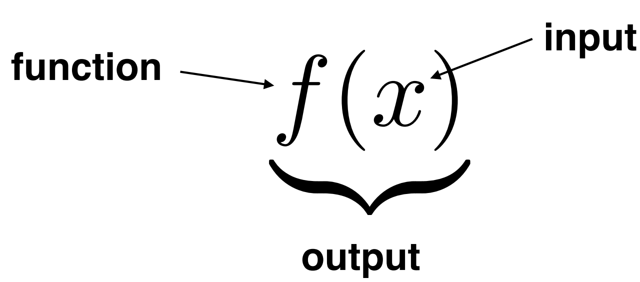


The Euclidean Space



Functions

A function is a mathematical machine
You input something
You get something out



As long as you input the same thing, you'll always get the same thing out.

Example Function

The exponential function: $\exp$

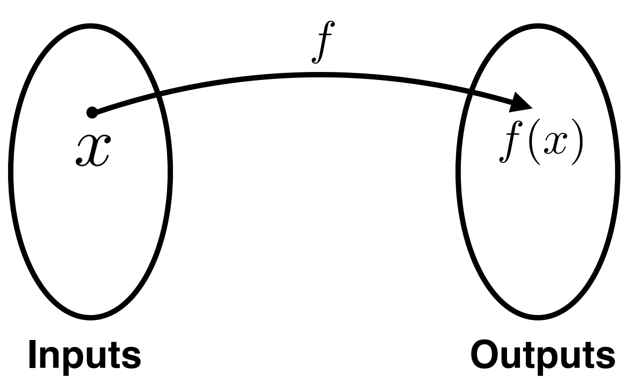
$$\exp(-1) = 0.3679$$

$$\exp(0) = 1$$

$$\exp(1) = 2.7183$$

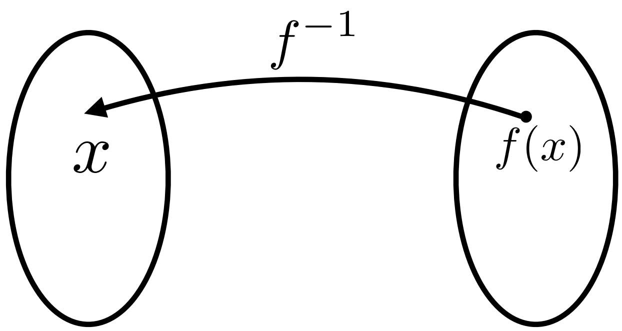
$$\exp(1.2) = 3.3201$$

A function maps inputs to outputs. It converts x to $f(x)$.



Inverse Function

The inverse function converts all $f(x)$ in Outputs back to x in Inputs.



Note: Not every function has an inverse function. An invertible function is a very special thing.

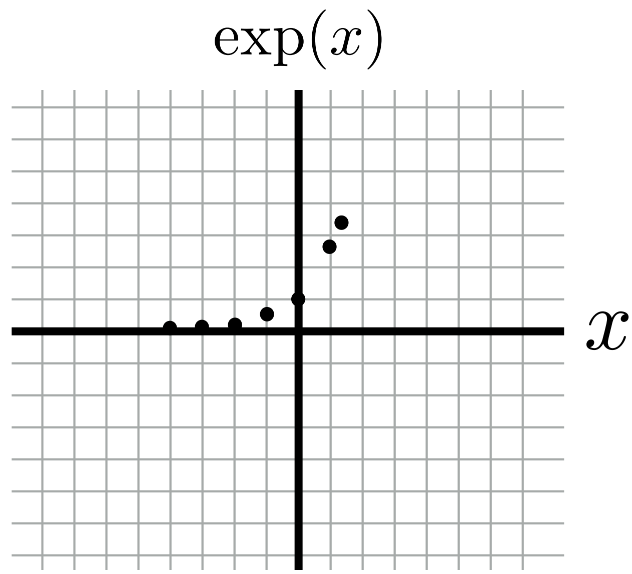
Example Inverse Function

The inverse of the $\exp$ function is the $\log$ function.

$\exp(-1) = 0.3679$	$\log(0.3679) = -1$
$\exp(0) = 1$	$\log(1) = 0$
$\exp(1) = 2.7183$	$\log(2.7183) = 1$
$\exp(1.2) = 3.3201$	$\log(3.3201) = 1.2$

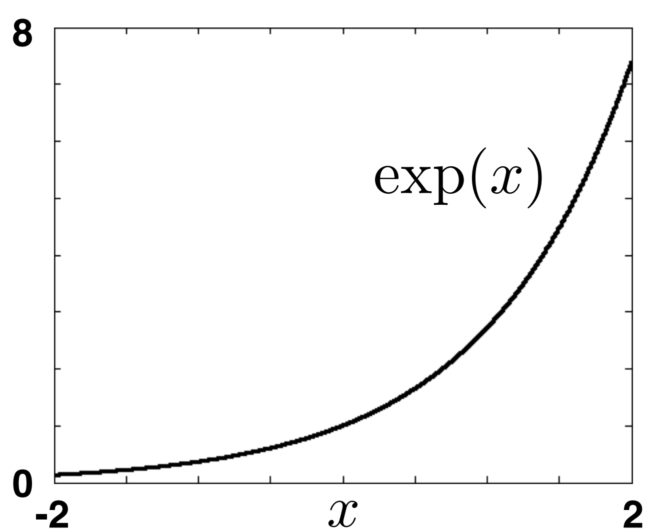
Graphing a Function

Showing the points of a function in a Euclidean Plane



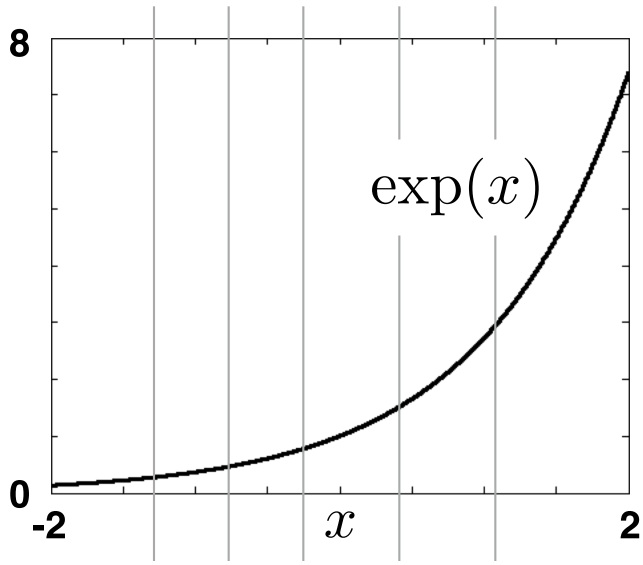
Graphing a Function

Showing all the points of a function in a Euclidean Plane



Vertical Line Test

If you draw a vertical line through the graph of a function, it will intersect at most one point.



Equation of a Line

One way to represent a line is with a function of the form

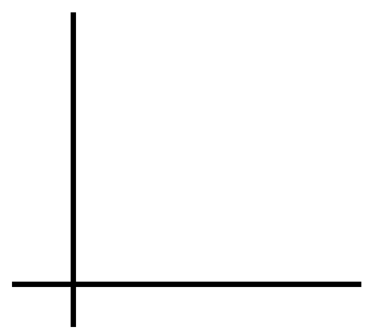
$$f(x) = m\,x + b$$

m **is called the "slope" of the line.**
b **is called the "vertical intercept" of the line.**

We can't represent vertical lines this way. We'll see a more general representation later.

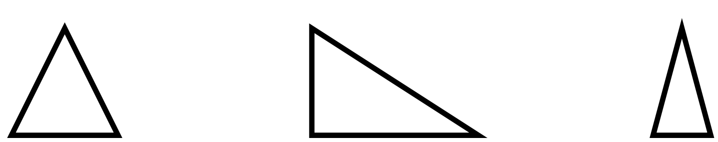
Perpendicular

Two lines are perpendicular means that the angle between them is $\pi/2$ radians.



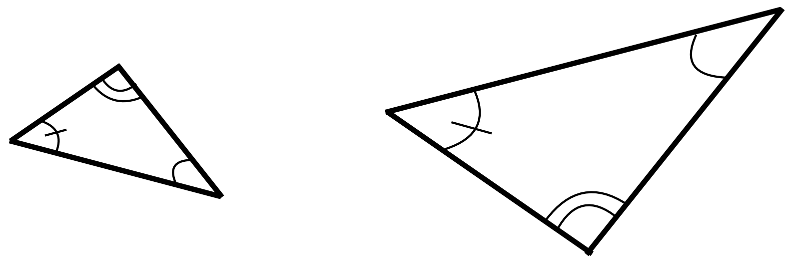
Triangles

A shape with three sides.

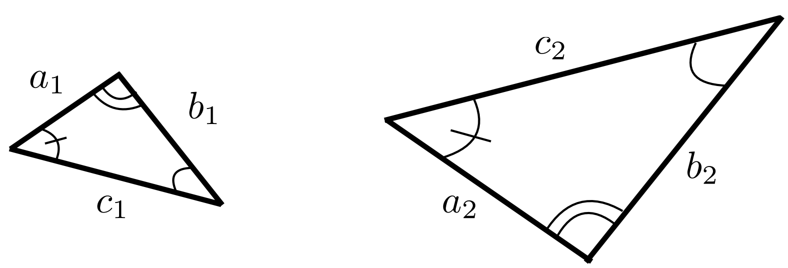


Similar Triangles

Two triangles that have the same angles are similar.



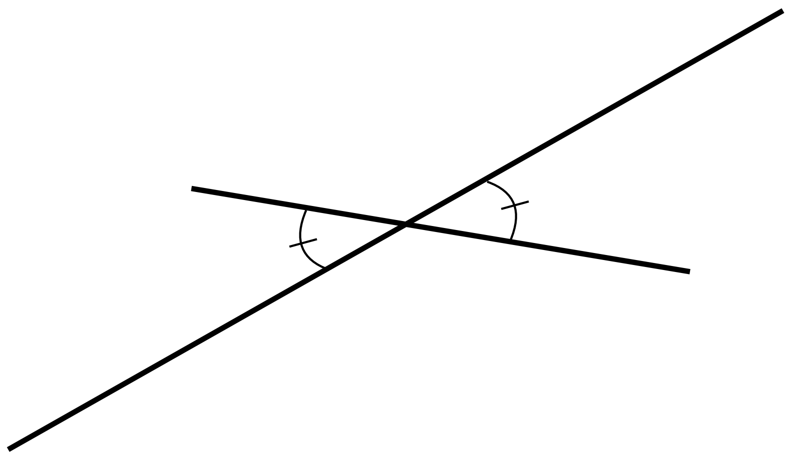
The ratios of equivalent lengths of similar triangles are equal



$$\frac{a_1}{b_1} = \frac{a_2}{b_2} \qquad \frac{b_1}{c_1} = \frac{b_2}{c_2} \qquad \frac{a_1}{c_1} = \frac{a_2}{c_2}$$

Equal and Opposite Angles

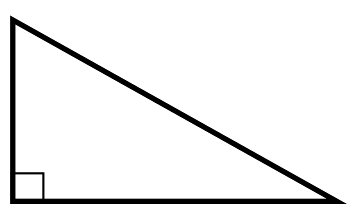
Opposite angles formed with straight lines are equal.



Right Triangle

A right triangle has one angle equal to $\pi/2$

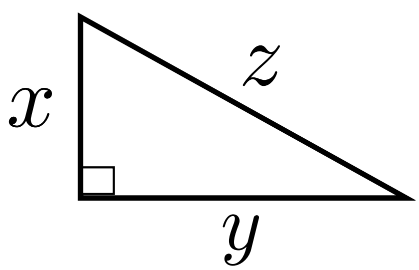
That angle is denoted with a square as shown



The line opposite the $\pi/2$ **angle is called the *hypotenuse*.**

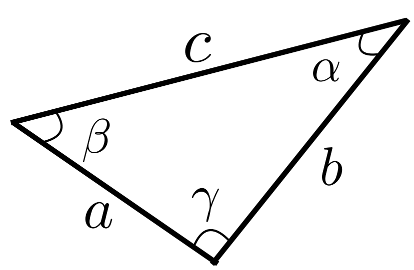
Pythagorean Theorem

The lengths of the sides of a right triangle satisfy the following formula



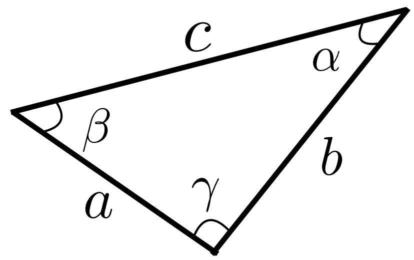
$$x^2 + y^2 = z^2$$

The three angles of a triangle add up to π



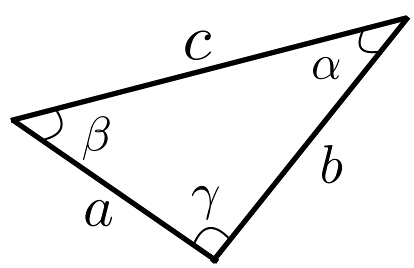
$$\alpha + \beta + \gamma = \pi$$

Law of Sines



$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$$

Law of Cosines

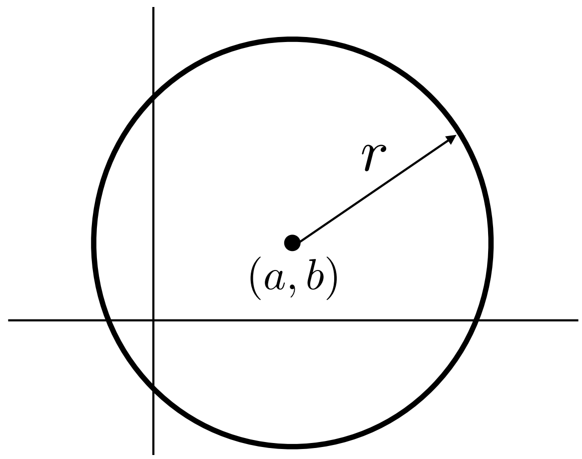


$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

This is a generalization of the Pythagorean Theorem.

Circle

A circle centered at *(a,b)* with radius *r* is the set of all points a distance *r* from *(a,b)*.



Equation of a Circle

Follows from the Pythagorean theorem.

$$(x-a)^2 + (y-b)^2 = r^2$$

The *center* of the circle is located at (a,b).

r **is called the circle's *radius.***

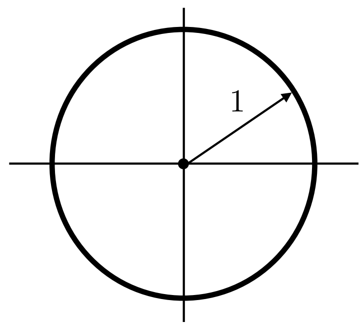
$2r$ **is called the *diameter* of the circle.**

The Unit Circle

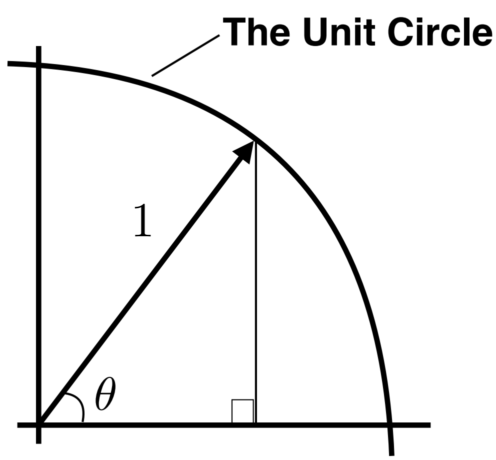
The unit circle is a circle of radius 1 centered at (0,0).

$$x^2 + y^2 = 1$$

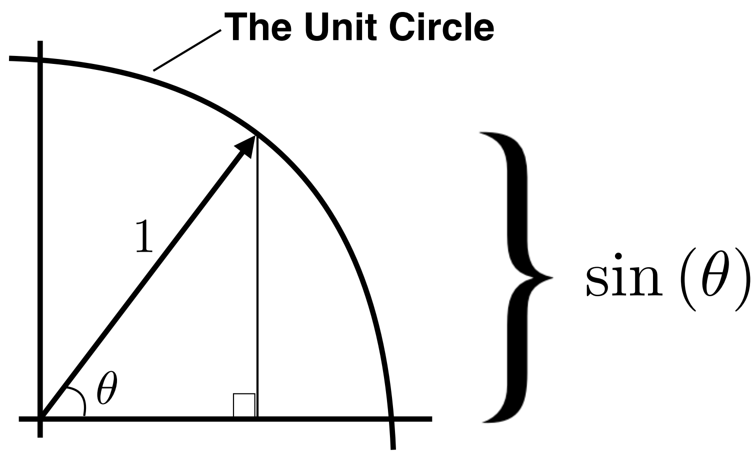
Graph of the unit circle



Each point on the unit circle makes a right triangle with a hypotenuse of length 1.

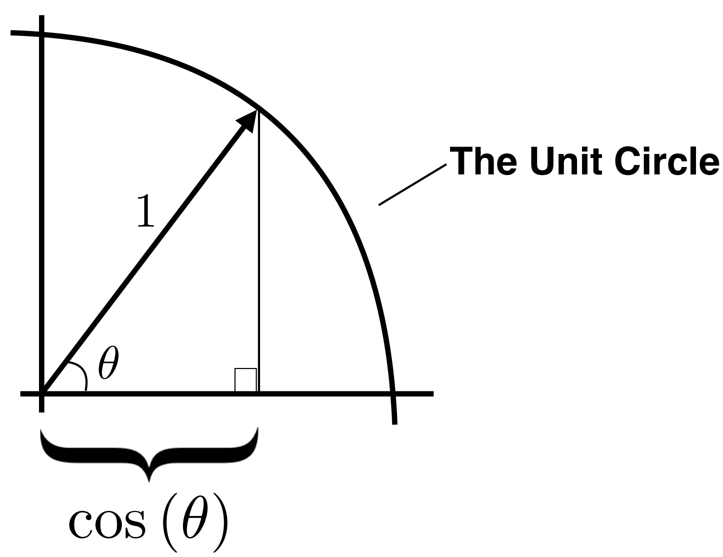


Sine



The Unit Circle
1
θ
$\sin(\theta)$

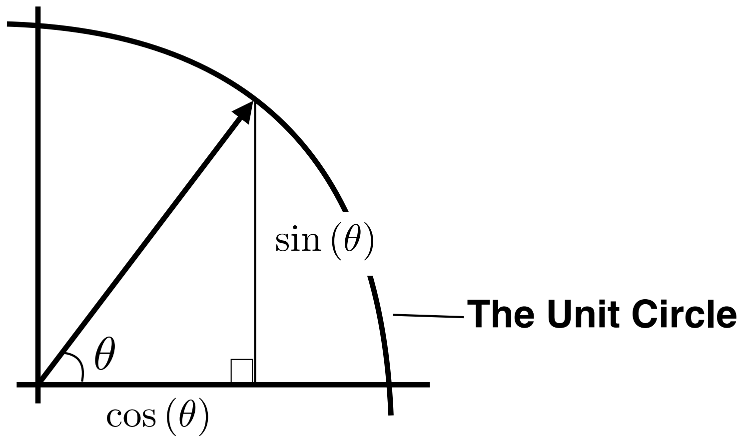
Cosine



The Unit Circle
1
θ
$\cos(\theta)$

From the Pythagorean Theorem:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$



The Number Pi - π

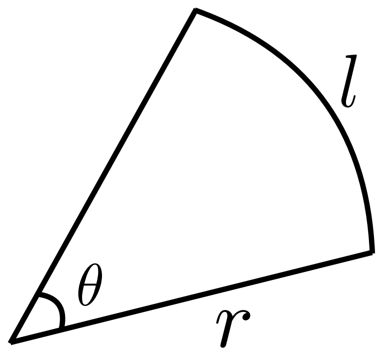
π is defined as

$$\pi = \frac{\text{Circumference of the Circle}}{\text{Diameter of a Circle}}$$

It turns out that this number is the same for all circles!

$$\pi \approx 3.141592653$$

Radians



$$l = r\,\theta$$

where l is the arc length
θ is in radians

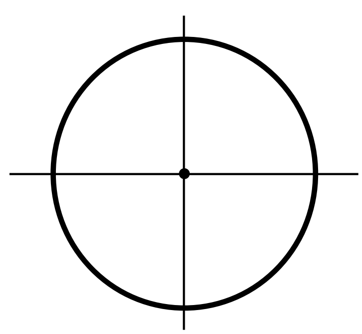
There are 2π radians in a full revolution.

There are π radians in a half revolution.

Degrees

There are 360° in a full revolution.

There are 180° in a half revolution.



We will not use degrees very much in this class.

Vectors

A vector is an ordered finite list of numbers.

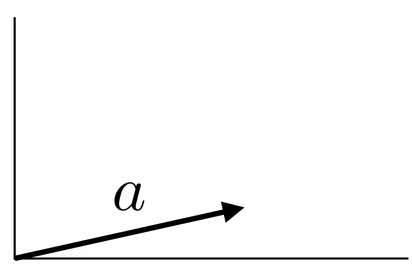
Example: $\begin{bmatrix} -1.1 \\ 0.0 \\ 3.6 \\ -7.2 \end{bmatrix}$ $(-1.1, 0.0, 3.6, -7.2)$

Example: 0

All the elements are 0.
The length is understood from context.

Drawing Vectors in 2D

$$a = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$



Vector Addition

Two vectors of the same size can be added together by adding corresponding components.

Example:

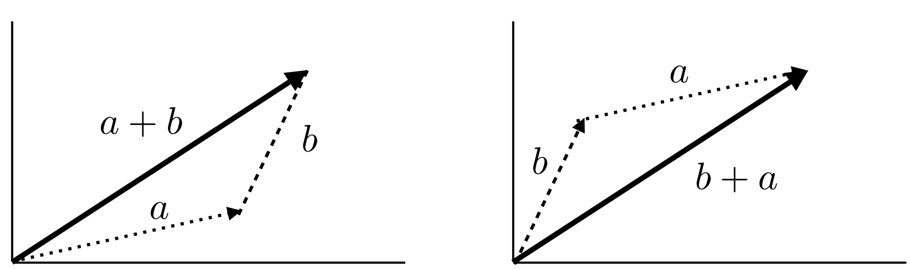
$$\begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 \\ 9 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

Geometric Interpretation

Vectors add tip-to-tail.



Scalar Multiplication

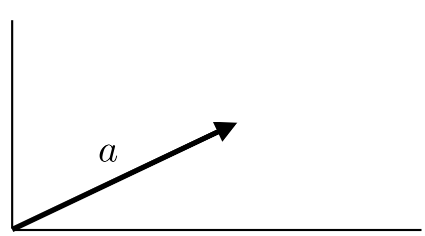
Every element of the vector is multiplied by the scalar (i.e. number)

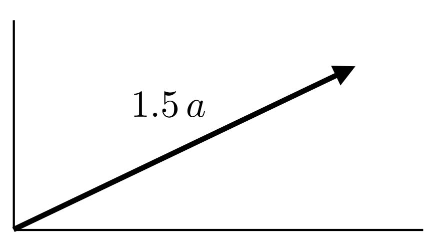
Example:

$$(-2)\begin{bmatrix} 1 \\ 9 \\ -6 \end{bmatrix} = \begin{bmatrix} -2 \\ -18 \\ 12 \end{bmatrix}$$

Geometric Interpretation

Vector is scaled by scalar multiplication.



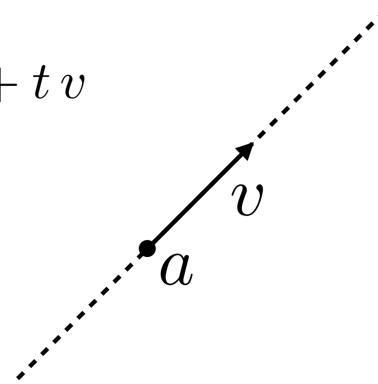


Equation of a Line

Suppose that a is a point on the line and v is a vector parallel to the line. The the line can be represented as

$$f(t) = a + t\,v$$

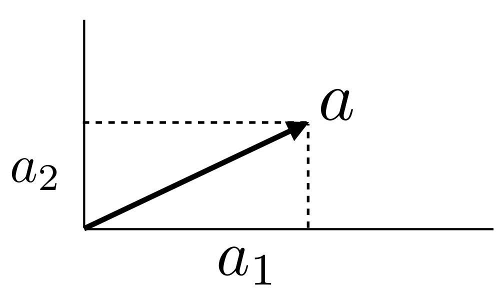
where t is any real number.



Length of a Vector

The length of a vector a, denoted by $\|a\|_2$, is

$$\|a\|_2 = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$



Dot Product

If a and b are vectors then

$$a \cdot b = a^T b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Perpendicular Vectors

Two vectors a and b are perpendicular if and only if

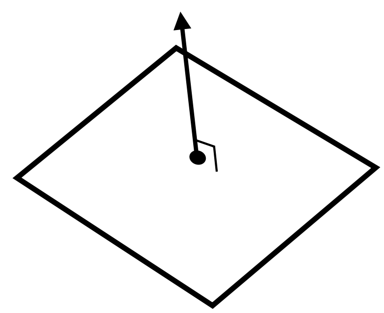
$$a \cdot b = 0$$

Equation of a Plane

A normal vector is a vector perpendicular to all points on a plane.

The equation of a plane is specified by a normal vector and a point on the plane.

The key is realizing how to write the equation of a plane using the dot product.

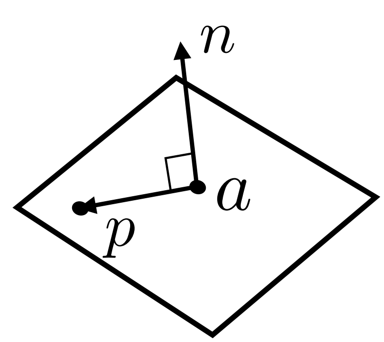


Equation of a Plane

Suppose n is the normal vector of a plane and a is a point that lies in the plane.

Then any other point p in the plane satisfies

$$n \cdot (p - a) = 0$$



Dot Product Properties

The angle between two vectors *a,b* **is acute if and only if**

$$a \cdot b > 0$$

The angle between two vectors *a,b* **is obtuse if and only if**

$$a \cdot b < 0$$

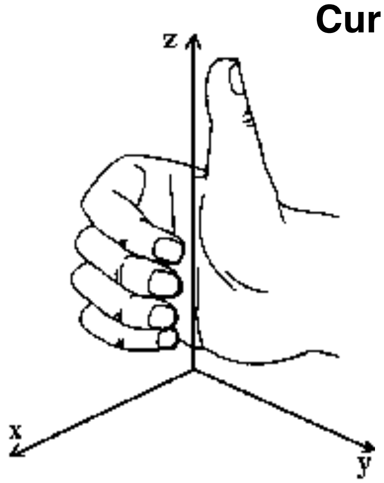
Cross Product

If a **and** b **are vectors with three elements then**

$$a \times b = \begin{bmatrix} a_2 b_3 - b_2 a_3 \\ b_1 a_3 - b_3 a_1 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$a \times b$ **is perpendicular to both** a **and** b **.**

Cross Product - Right Hand Rule



Curl fingers from vector x to y.

Thumb points in direction of

$$x \times y$$

Cross Product and Parallel Vectors

The cross product of parallel vectors is 0.

Size of a Vector

If a is a vector then its size is

$$\|a\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Note: unless otherwise specified $\|a\| = \|a\|_2$.

Linear Combination

Suppose $a_1, a_2, \ldots, a_n$ are vectors of the same size.

A linear combination of these vectors is an expression of the form

$$\beta_1\, a_1 + \beta_2\, a_2 + \cdots + \beta_n\, a_n$$

where $\beta_1, \beta_2, \ldots, \beta_n$ are numbers.

Span

Suppose $a_1, a_2, \ldots, a_n$ **are vectors of the same size.**

The span of $\{a_1, a_2, \ldots, a_n\}$ **is the set of *all* linear combinations of the vectors in the set.**

Matrices

A matrix is a rectangular array of numbers.

Example:

$$\begin{bmatrix} 0 & 1 & -2.3 & 0.1 \\ 1.3 & 4 & -0.1 & 0 \\ 4.1 & -1 & 0 & 1.7 \end{bmatrix}$$

This matrix has 3 rows and 4 columns. We call it a 3x4 matrix.

A matrix with the same number of rows and columns is called a square matrix.

Matrix Transpose

The transpose of the matrix is the result of flipping the matrix about its diagonal.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & & a_{MN} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{M1} \\ a_{12} & a_{22} & & a_{M2} \\ \vdots & & \ddots & \vdots \\ a_{1N} & a_{2N} & & a_{MN} \end{bmatrix}$$

Matrix-Scalar Multiplication

$$k \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} = \begin{bmatrix} k\,a_{11} & k\,a_{12} & \cdots & k\,a_{1N} \\ k\,a_{21} & k\,a_{22} & \cdots & k\,a_{2N} \\ \vdots & & \ddots & \vdots \\ k\,a_{M1} & k\,a_{M2} & \cdots & k\,a_{MN} \end{bmatrix}$$

Each element of the matrix is multiplied by the scalar.

Matrix-Vector Multiplication

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} v = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1N} \end{bmatrix} v_1 + \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2N} \end{bmatrix} v_2 + \cdots + \begin{bmatrix} a_{M1} \\ a_{M2} \\ \vdots \\ a_{MN} \end{bmatrix} v_N$$

The result is a linear combination of the columns of the matrix.
The linear coefficients are the elements of the vector.

Matrix-Vector Multiplication

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} v = \begin{bmatrix} — & r_{a,1}^T & — \\ — & r_{a,2}^T & — \\ & \vdots & \\ — & r_{a,M}^T & — \end{bmatrix} v = \begin{bmatrix} r_{a,1}^T v \\ r_{a,1}^T v \\ \vdots \\ r_{a,M}^T v \end{bmatrix}$$

Each element of the result is the dot product of the rows of the matrix with the vector.

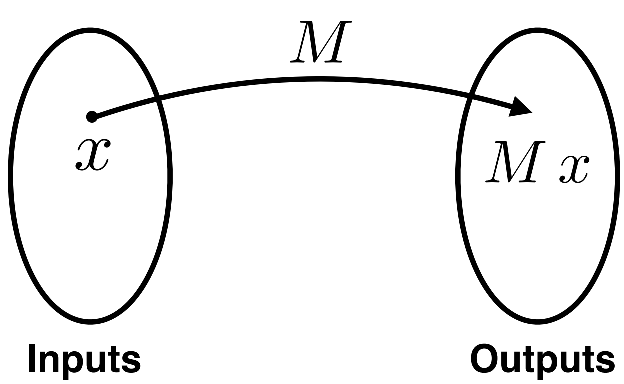
Identity Matrix

The Identity Matrix is a matrix with 1s along the diagonal and zeros everywhere else.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ & & & 1 & 0 \\ 0 & & \cdots & 0 & 1 \end{bmatrix}$$

Question: What is Iv for any vector v ?

Matrix multiplication is a type of function.



Matrix-Matrix Multiplication

$$U \underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_N \\ | & | & & | \end{bmatrix}}_{V} = \begin{bmatrix} | & | & & | \\ U\,v_1 & U\,v_2 & \cdots & U\,v_N \\ | & | & & | \end{bmatrix}$$

Each column of the output is the result of the matrix U times the corresponding column of the matrix V.

Matrix-Matrix Multiplication

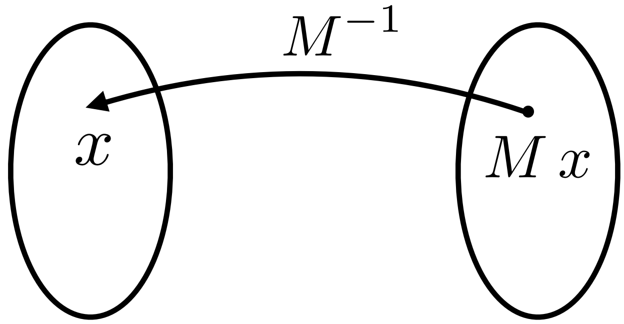
$$\begin{bmatrix} — & r_{u,1}^T & — \\ — & r_{u,2}^T & — \\ & \vdots & \\ — & r_{u,M}^T & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ c_{v,1} & c_{v,2} & \cdots & c_{v,N} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} r_{u,1}^T c_{v,1} & r_{u,1}^T c_{v,2} & \dots & r_{u,1}^T c_{v,N} \\ r_{u,2}^T c_{v,1} & r_{u,2}^T c_{v,2} & \dots & r_{u,2}^T c_{v,N} \\ \vdots & & \ddots & \vdots \\ r_{u,M}^T c_{v,1} & r_{u,M}^T c_{v,2} & \dots & r_{u,M}^T c_{v,N} \end{bmatrix}$$

Each element of the output is a dot product of the rows of the first matrix with the columns of the second.

Matrix Inverse

For some matrices, there exists an inverse matrix such that

$$M^{-1}M = I$$



Note: it's a very special thing for a matrix to be invertible.

Theorem

Only square matrices can be invertible.

Block Matrix

A matrix where each element is a matrix.

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & & A_{2N} \\ \vdots & & \ddots & \vdots \\ A_{M1} & A_{M2} & & A_{MN} \end{bmatrix}$$

Here, each A_{ij} **is a matrix.**

Block Matrix Multiplication

Block Matrix Multiplication is just like the dot product matrix multiplication.

$$\begin{bmatrix} - & r_{A,1}^T & - \\ - & r_{A,2}^T & - \\ & \vdots & \\ - & r_{A,M}^T & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ c_{B,1} & c_{B,2} & \cdots & c_{B,N} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} r_{A,1}^T c_{B,1} & r_{A,1}^T c_{B,2} & \cdots & r_{A,1}^T c_{B,N} \\ r_{A,2}^T c_{B,1} & r_{A,2}^T c_{B,2} & \cdots & r_{A,2}^T c_{B,N} \\ \vdots & & \ddots & \vdots \\ r_{A,M}^T c_{B,1} & r_{A,M}^T c_{B,2} & \cdots & r_{A,M}^T c_{B,N} \end{bmatrix}$$