Initial Applications

Math Lecture 2

Nicholas Dwork

Systems of Equations

Suppose we have multiple equations:

$$\begin{aligned} 2x - 5y &= 8 \\ 3x + 9y &= -12 \end{aligned}$$

We can rewrite these equations as a single equation using matrices.

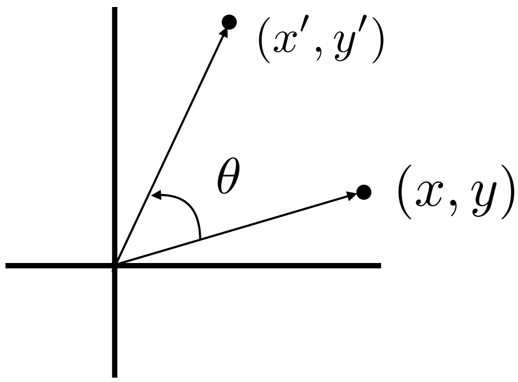
$$\begin{bmatrix} 2 & -5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -12 \end{bmatrix}$$

It then becomes easier to solve!

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 3 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -12 \end{bmatrix}$$

2D Rotation Matrix

Rotating a point about the origin by angle θ:



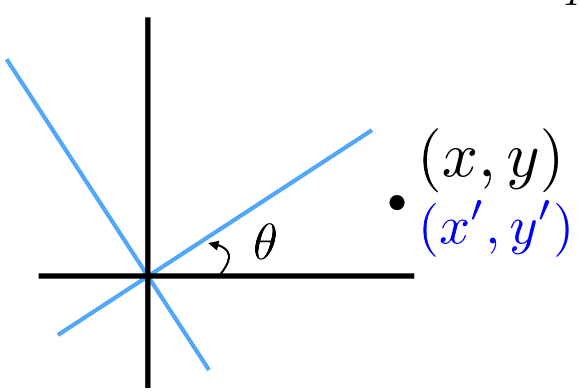
$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R(\theta) \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Rotation Matrix

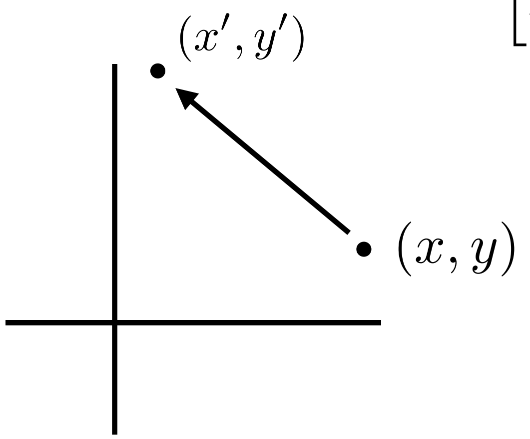
Rotating the axis by angle θ:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R(\theta) \begin{bmatrix} x \\ y \end{bmatrix}$$

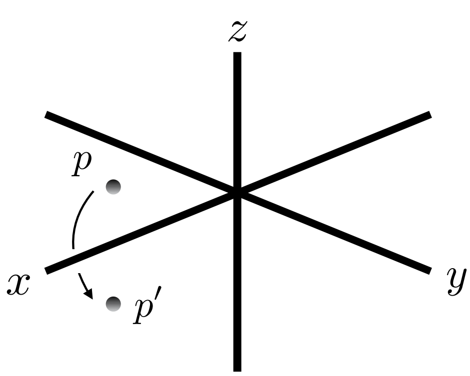
2D Translation



$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

3D Rotation X Matrix

A rotation of a point about the x axis.



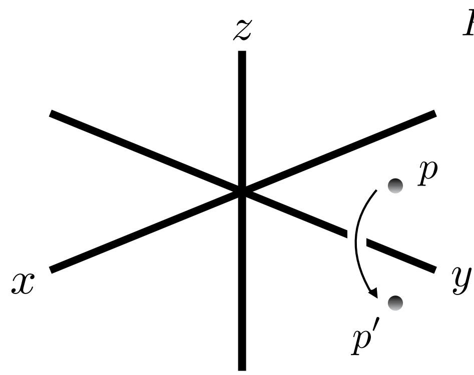
$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = R_x(\theta) \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$p' = R_x(\theta) p$$

3D Rotation Y Matrix

A rotation of a point about the y axis.



$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = R_y(\theta) \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$p' = R_y(\theta)p$$

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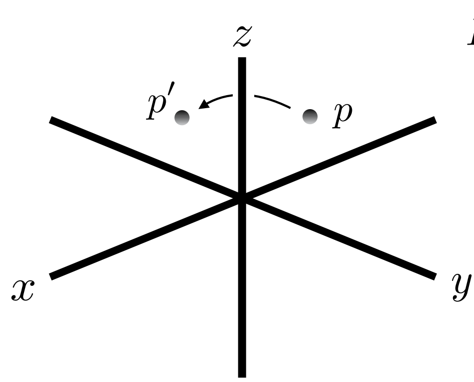
3D Rotation Z Matrix

A rotation of a point about the z axis.

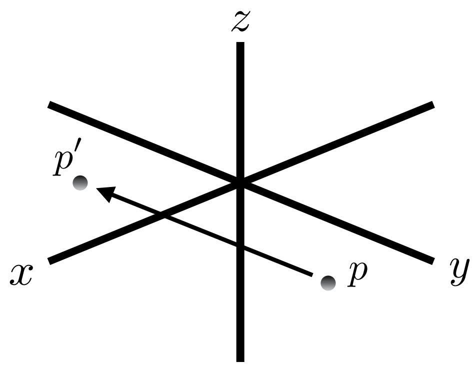


$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = R_z(\theta) \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$p' = R_z(\theta)p$$

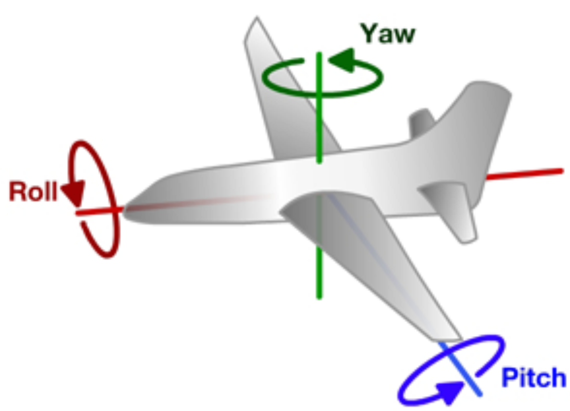
3D Translation



$$p + \Delta p = p'$$

$$\Delta p = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$$

Roll, Pitch, and Yaw

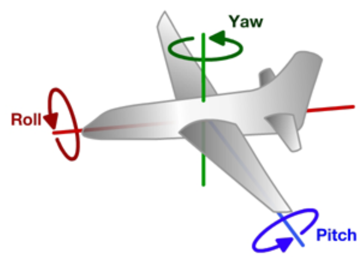


http://theboredengineers.com/2012/05/the-quadcopter-basics/

Note: rotations are defined with respect to the object

Since matrices don't commute, it's not enough to know which axis roll, pitch, and yaw correspond to.

You must also know which order they are applied.

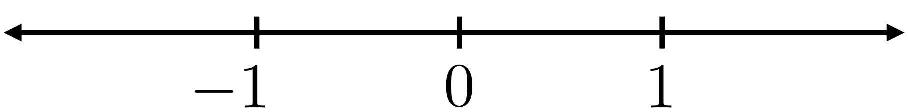


http://theboredengineers.com/2012/05/the-quadcopter-basics/

Two common conventions:

- **Roll - Pitch - Yaw**
- **Yaw - Pitch - Roll**

Real Number Line



Question: How do we represent the value infinity as a real number?

Answer: you can't. Infinity is not a real number.

Homogeneous Numbers

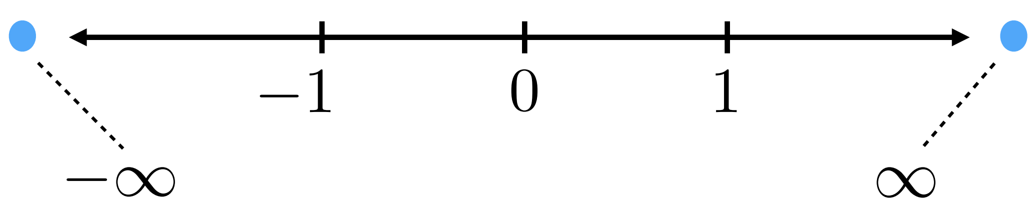
Each value is represented using two real numbers.

Homogeneous	Euclidean
(x, w)	$\frac{x}{w}$
$(8, 1)$	8
$(16, 2)$	8
$(2\pi, 0.5)$	4π
$(3, 0)$	No equivalent

The representation of a value is no longer unique.

We've added two points that we previously didn't have:
We call these the points at infinity.

Homogeneous Line



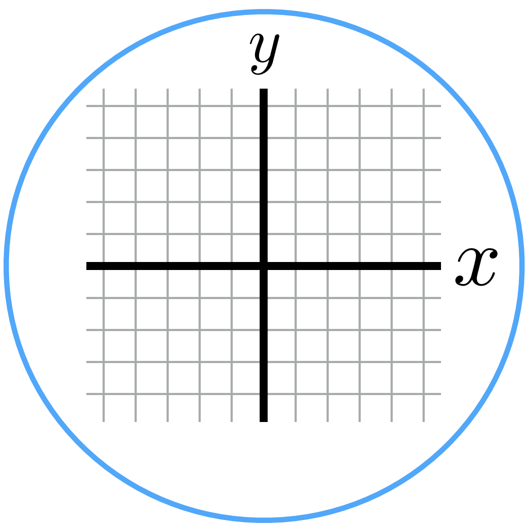
2D Homogeneous Coordinates

Each coordinate is represented using 3 real numbers.

Homogeneous	**Euclidean**
(x, y, w)	$\left(\frac{x}{w}, \frac{y}{w}\right)$
$(1, 2, 3)$	$\left(\frac{1}{3}, \frac{2}{3}\right)$
$(8, -2, 0)$	**No equivalent**

There exist a set of points (a,b,0) that do not exist in Euclidean coordinates.
We call this set “The line at infinity”.

The Homogeneous Plane



It’s like we’ve added a ring around the Euclidean plane.

3D Homogeneous Coordinates

Each coordinate is represented using 4 real numbers.

Homogeneous	Euclidean
(x, y, z, w)	$\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right)$
$(x, y, z, 0)$	No equivalent

Homography

A square matrix applied to homogeneous coordinates.