Digital Signals
Assignment 2: Due 6/28/2016

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1 Main Problems

Problem 1 A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear means that

- $f(x+y) = f(x) + f(y)$

- $f(\alpha x) = \alpha f(x)$

for any vectors x and y, and scalar α. Show that for any matrix A, $f(x) = A x$ is a linear function.

Problem 2 (*Credit: Dr. Boyd*)   Consider vector x. Either prove the following statement is true or provide a counter example:

$$\text{mean}\,(\alpha\, x + \beta 1) = \alpha\, \text{mean}(x) + \beta.$$

Note: here 1 is the vector where each element has value 1 (this is unambiguous due to the context of the problem).

Problem 3 *Commutativity of Matrices*
  For this problem, consider matrices with real entries.

  Part a Either prove that matrices commute with addition or give a counter example.

  Part b Either prove that matrices commute with multiplication or give a counter example.

Problem 4 (*Credit: Dr. Boyd*) Consider the rational function (i.e., a ratio of polynomials)

$$f(t) = \frac{c_0 + c_1 t + c_2 t^2}{1 + d_1 t + d_2 t^2}$$

that statisfies the following conditions:

$$f(1) = 2, \quad f(2) = 5, \quad f(3) = 9, \quad f(4) = -2, \quad f(5) = -4.$$

Find the numbers c_0, c_1, c_2, d_1, and d_2 for which these conditions hold.

  Part a Let $x = (c_0, c_1, c_2, d_1, d_2)$. Explain how to formulate this problem as finding a vector x that satisfies a system of linear equations $Ax = b$. Be sure to specify what A and b are.

Part b Solve the system of linear equations with Python. What do you get for vector x? Plot the function f for `x=np.linspace(-2,8,50)`.

Problem 5 Read some part of Reality is Broken. Describe how it adheres to and/or conflicts with conventional thought. What is your opinion of what you've read? (Please state which pages you read.)

Problem 6 Create a matrix of zeros 500×500 in size called "myImage". Then execute the following command:

```
myImage[1:100,1:200] = 255
```

Display this image on the screen. What does this show you about image coordinates in Matlab? (Provide as many conclusions as you can.)

Some sample code showing how to read in and display an image can be found here:

`http://web.stanford.edu/~ndwork/si2016/hmwk2/showImage.py`

Problem 7 There is a way to embed a message in an image. Very soon, some students will find you; they will be relying on you to decode a message for them so that they can progress in their game. They will present you with an image and a matrix and ask you for the message. The fate of the world rests on your shoulders.

Let X denote the original image. The variable x is a vector formed by column extending the image; it has N elements. We will alter it as follows: $y = x + A s$, where A is a matrix and s is the message. The matrix A is of size $N \times S$, where S is the number of symbols in the message. We assume that the message is binary code, so s is a vector only of values 0 or 1. The entries of A are small so that $y \approx x$.

To recover s, we use the pseudo-inverse of the matrix A, denoted as $A^\dagger$. Let

$$\tilde{s} = A^\dagger\, y = A^\dagger\, (x + As) = A^\dagger\, x + A^\dagger As = A^\dagger\, x + s.$$

The value $A^\dagger\, x$ is noise, and we hope that it's small compared to our message s. We form our estimate of the code as $\hat{s} = \text{round}(\tilde{s})$.

Write a code that extracts the image from the matrix. Your function prototype should be as follows:

```
def extractMessage( img, A )
  # relevant code goes here
  return message
```

There is a sample image to do this with at

`www.stanford.edu/~ndwork/si2016/hmwk2/embedded.png`

The matrix A that encoded the message is in the following file:

`www.stanford.edu/~ndwork/si2016/hmwk2/encodeMatrix.npy`

I've written some skeleton code for you to read in the matrix and the image from the file. You need to figure out how to decode the message. You can get the skeleton code here:

```
www.stanford.edu/~ndwork/si2016/hmwk2/decodeMessage.py
```

The message has been represented using Huffman encoding. (If I've forgotten to explain this to you, please remind me so that I can teach this in class.) You can find the Huffman encoding tree here:

```
www.stanford.edu/~ndwork/si2016/hmwk2/huffCode.png
```

What message do you get out?

Problem 8   Let $x, y \in \mathbb{R}^3$. Show $x^T y = y^T x$.

Problem 9   (*Credit: Dr. Boyd*) In this puzzle we consider several linear functions of a grayscale image with $N \times N$ pixels. To keep the matrices small enough to work out by hand, we will consider the case when $N = 3$ (which would hardly qualify as an image). We represent a 3×3 image as a vector in $\mathbb{R}^9$ using the ordering of pixels shown below. (This is called *column-major* ordering.)

$$
\begin{array}{|c|c|c|}
\hline
X_1 & X_4 & X_7 \\
\hline
X_2 & X_5 & X_8 \\
\hline
X_3 & X_6 & X_9 \\
\hline
\end{array}
$$

Each of the operations or transformations below defines a function $y = f(x)$, where x is the original 9-vector, and y is the 9-vector that corresponds to the transformed image. For each of the following operations, give the 9×9 matrix A for which $y = A x$.

Part a   Turn the original image upside down.

Part b   Rotate the original image clockwise by $\frac{\pi}{2}$ radians.

Part c   Translate the image up by 1 pixel and right by 1 pixel. In the translated image, assign the value $y_i = 0$ to the pixels in the first column and the last row.

Part d   Set each pixel y_i to be the average of the neighbors of pixel i in the original image. By neighbors, we mean the pixels immediately above and below, and immediately to the left and right. The center pixel has 4 neighbors, corner pixels have two neighbors, and the remaining pixels have 3 neighbors.

Problem 10   Recall that a field of math is defined by a set of axioms. Axioms are statements that we accept as true without proof. The axioms of the real numbers are as follows:

- There exists a number 0 such that for any number a, $a + 0 = a$.

- There exists a number 1 such that for any number 1, $a \cdot 1 = a$.

- For any number a, there exists a number $(-a)$ such that $a + (-a) = 0$

- For any number $a \neq 0$, there exists a number a^{-1} such that $a \cdot a^{-1} = 1$.

- For any two numbers a, b, $a + b = b + a$.

- For any two numbers a, b, $ab = ba$.

- For any three numbers a, b, c, $(a + b) + c = a + (b + c)$.

- For any three numbers a, b, c, $(ab)c = a(bc)$.

- For any numbers a, b, c, $a(b + c) = ab + ac$.

A theorem is a statement that we can prove given the axioms. Prove the following theorem:

- For any number a, $a \cdot 0 = 0$.

Problem 11 Consider a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$. There exists a matrix A such that $f(x) = A\,x$ for all x.

   Part a What size is A?

   Part b Find A (in terms of f applied to specific vectors). *Hint: consider $f\left((1, 0, 0, \ldots, 0)\right)$.*

Problem 12 Consider a matrix $A \in \mathbb{R}^{M \times N}$. Prove that the $\mathrm{Im}(A)$ and $\mathrm{Null}(A)$ are vector spaces.

2 Challenge Problems

CP 1 Let a and b be the following vectors:

$$a = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 8 \end{bmatrix}.$$

The vector b can be written as the sum of two vectors, one that's parallel to a and one that's perpendicular to a. Find these vectors. (Note: The vector that is parallel to b is called "the projection of a onto b".)

CP 2 The reverse triangle inequality states that for any two vectors a and b,

$$\left| \|a\|_2 - \|b\|_2 \right| \leq \|a - b\|_2.$$

Prove this.

CP 3 The set of polynomials of order 4 is a vector space.

   Part a Prove that this set is a vector space.

   Part b Find a basis for this vector space.

CP 4 Write a function that accepts a natural number and outputs a triangle of numbers so that the elements in each column indicate the reverse column id. The prototype of the function should be as follows:

```
function makeNumberTriangle( N )
```

A sample output where the input number $N = 4$ is shown below

```
            1
         2  1
      3  2  1
   4  3  2  1
```

Make sure the function dies nicely if someone inputs something other than a natural number (look at the error function in Matlab).