Vector Spaces

Math Lecture 3

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Binary Operation

Given a set S, a Binary Operation is a rule for mapping pairs of elements of S to another element of S.

Vector Space

A vector space over a field F is a set S with binary operations + and x that satisfy the following:

There exists an element 0 such that u + 0 = u
u+v = v+u
(u+v)+w = u+(v+w)
For any u there exists -u such that u + (-u) = 0

There exists a scalar 1 such that 1 x u = u

For any scalar k, k (u + v) = ku + kv
For any scalars k1 and k2, (k1 + k2) u = k1 u + k2 u

Examples of Vector Spaces

$\mathbb{R}^2$ **Euclidean Plane**

$\mathbb{R}^3$ **Euclidean Space**

$\mathbb{R}^{3000}$

The set of all functions

The set of all continuous functions

The set of all polynomials of order 4.

Subspace

A subset of a vector space that is also a vector space is called a subspace.

To show that a subset of a Vector Space is a subspace, one must show:

- **It contains the 0 vector**
- **It is closed under scalar multiplication**
- **It is closed under vector addition**

Example:

Any plane through the origin is a subspace of $\mathbb{R}^3$

Basis

A Basis for a Vector Space V is a set of linearly independent vectors that spans the space.

The standard basis is denoted by e

For $\mathbb{R}^3$**, the standard basis is** $e = \{e_1, e_2, e_3\}$**.**

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Dimension

The size of any basis for a Vector Space V is the same.

The size of the basis is called the Vector Space's Dimension.

A Vector Space may not have a finite dimension.

Projection Onto Vector Space

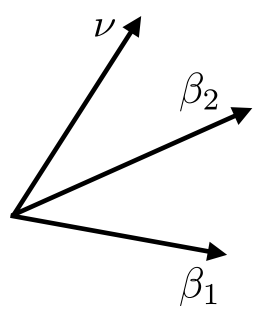
We've already discussed how to project a vector onto another vector.

Now we'll project a vector onto a Vector Space.

Projection Onto Vector Space

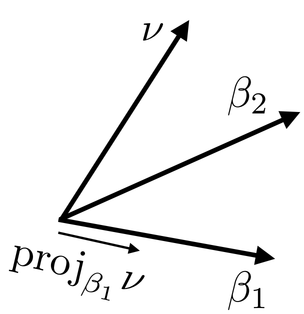
Let $\beta = \{\beta_1, \beta_2, \ldots, \beta_N\}$ **be a basis for the Vector Space** *V***.**

We would like to project ν **onto** *V.*



Projection Onto Vector Space

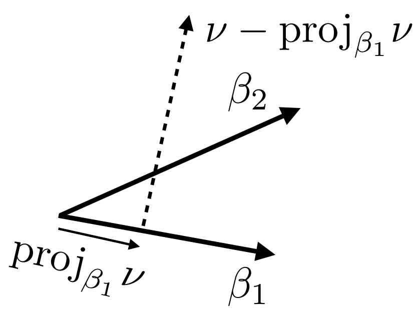
First, project ν **onto** β_1**.**



Projection Onto Vector Space

First, project ν onto β_1.

Subtract $\text{proj}_{\beta_1}\nu$ from ν.



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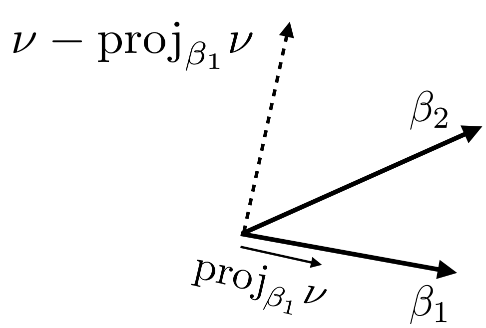
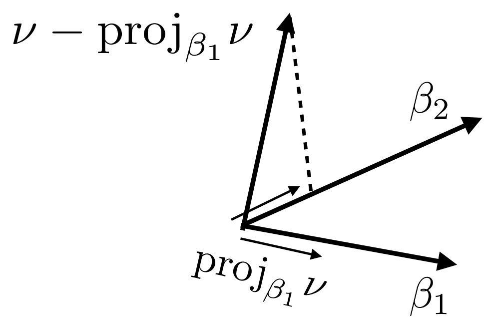


Projection Onto Vector Space

First, project ν onto β_1.

Subtract $\mathrm{proj}_{\beta_1}\nu$ from ν.

Project $\nu - \mathrm{proj}_{\beta_1}\nu$ onto β_2.



Projection Onto Vector Space

The projection of ν onto the Vector Space V is

$$\mathrm{proj}_{\beta_1}\nu + \mathrm{proj}_{\beta_2}\left(\nu - \mathrm{proj}_{\beta_1}\nu\right)$$