Matrices
Math Lecture 4

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Matrices

A matrix is a rectangular array of numbers.

Example:

$$\begin{bmatrix} 0 & 1 & -2.3 & 0.1 \\ 1.3 & 4 & -0.1 & 0 \\ 4.1 & -1 & 0 & 1.7 \end{bmatrix}$$

This matrix has 3 rows and 4 columns. We call it a 3x4 matrix.

A matrix with the same number of rows and columns is called a square matrix.

Matrix Transpose

The transpose of the matrix is the result of flipping the matrix about its diagonal.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & & a_{MN} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{M1} \\ a_{12} & a_{22} & & a_{M2} \\ \vdots & & \ddots & \vdots \\ a_{1N} & a_{2N} & & a_{MN} \end{bmatrix}$$

Matrix-Scalar Multiplication

$$k \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} = \begin{bmatrix} k\,a_{11} & k\,a_{12} & \cdots & k\,a_{1N} \\ k\,a_{21} & k\,a_{22} & \cdots & k\,a_{2N} \\ \vdots & & \ddots & \vdots \\ k\,a_{M1} & k\,a_{M2} & \cdots & k\,a_{MN} \end{bmatrix}$$

Each element of the matrix is multiplied by the scalar.

Matrix-Vector Multiplication

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} v = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1N} \end{bmatrix} v_1 + \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2N} \end{bmatrix} v_2 + \cdots + \begin{bmatrix} a_{M1} \\ a_{M2} \\ \vdots \\ a_{MN} \end{bmatrix} v_N$$

The result is a linear combination of the columns of the matrix.
The linear coefficients are the elements of the vector.

Matrix-Vector Multiplication

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} v = \begin{bmatrix} — & r_{a,1}^T & — \\ — & r_{a,2}^T & — \\ & \vdots & \\ — & r_{a,M}^T & — \end{bmatrix} v = \begin{bmatrix} r_{a,1}^T v \\ r_{a,1}^T v \\ \vdots \\ r_{a,M}^T v \end{bmatrix}$$

Each element of the result is the dot product of the rows of the matrix with the vector.

Identity Matrix

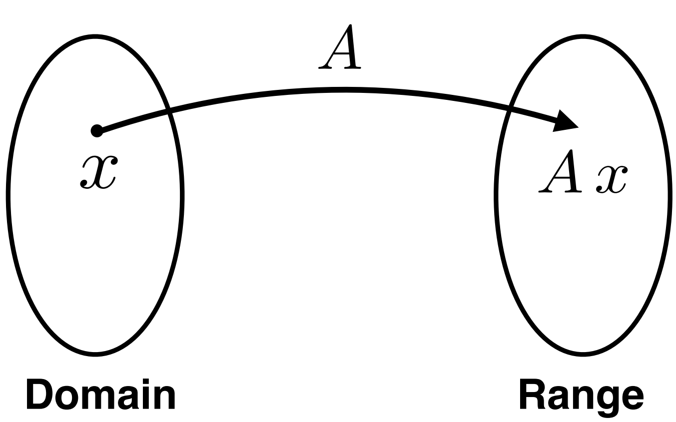
The Identity Matrix is a matrix with 1s along the diagonal and zeros everywhere else.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ & & & 1 & 0 \\ 0 & & \cdots & 0 & 1 \end{bmatrix}$$

Question: What is Iv for any vector v ?

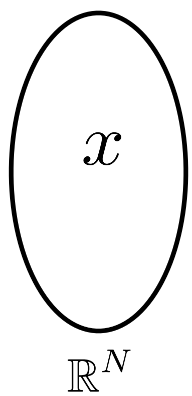
Consider an $M \times N$ Matrix A with real entries.

Matrix-vector multiplication with matrix A is a function!



Consider an $M \times N$ Matrix A with real entries.

Matrix A can only multiply vectors from $\mathbb{R}^N$. So that's its Domain.



Consider an $M \times N$ Matrix A with real entries.

Multiplication by A can only output vectors from $\mathbb{R}^M$ so that's its Range.

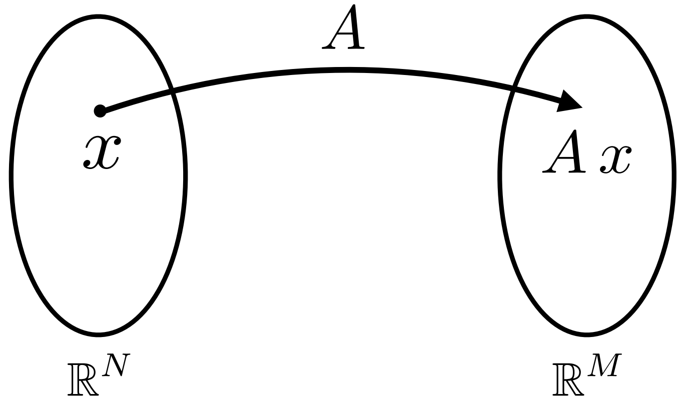
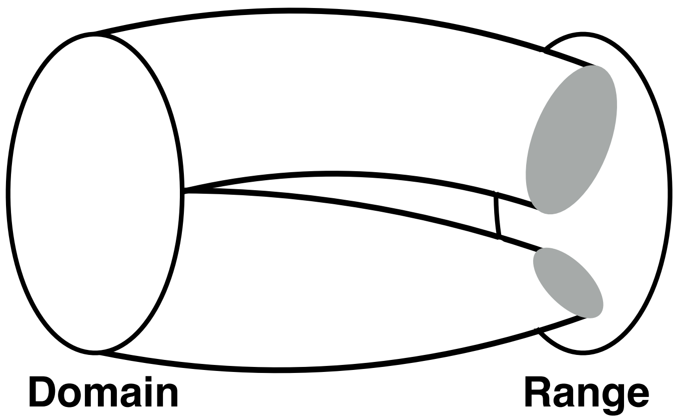


Image of a Matrix

The set of all $M\,x$ is called the Image of M.

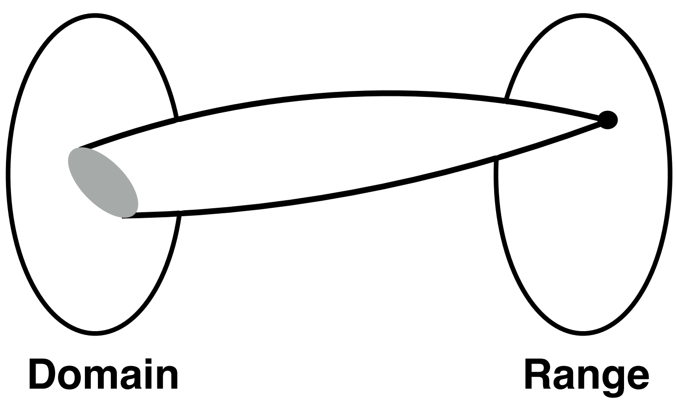
The Image of a matrix is a subset of the Range.



Pre-image of a Matrix

The set of all inputs that map to a value is called the pre-image of that value for the Matrix.

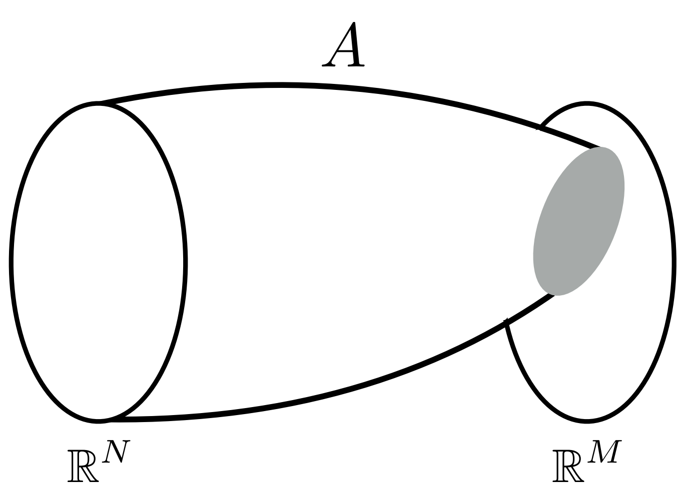
The Pre-image of a value is a subset of the Domain.



$$\text{Im}(M) = \left\{ Mx \text{ for all } x \in \mathbb{R}^M \right\}$$

$$\text{Pre-image}(y) = \{x \text{ such that } Mx = y\}$$

Question:
What is the Image of Matrix *A*?



Matrix-Vector Multiplication

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} v = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1N} \end{bmatrix} v_1 + \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2N} \end{bmatrix} v_2 + \cdots + \begin{bmatrix} a_{M1} \\ a_{M2} \\ \vdots \\ a_{MN} \end{bmatrix} v_N$$

The result is a linear combination of the columns of the matrix.
The linear coefficients are the elements of the vector.

Image of a Matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} v = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1N} \end{bmatrix} v_1 + \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2N} \end{bmatrix} v_2 + \cdots + \begin{bmatrix} a_{M1} \\ a_{M2} \\ \vdots \\ a_{MN} \end{bmatrix} v_N$$

From the definition of Matrix-Vector multiplication, we see that the Image of a Matrix is the span of its columns.

Another name for the Image of a Matrix is “column space”.

Image of a Matrix

$$\mathrm{Im}(A) = \mathrm{span}(c_1, c_2, \cdots, c_N)$$

where c_i is the i^{th} column of a matrix.

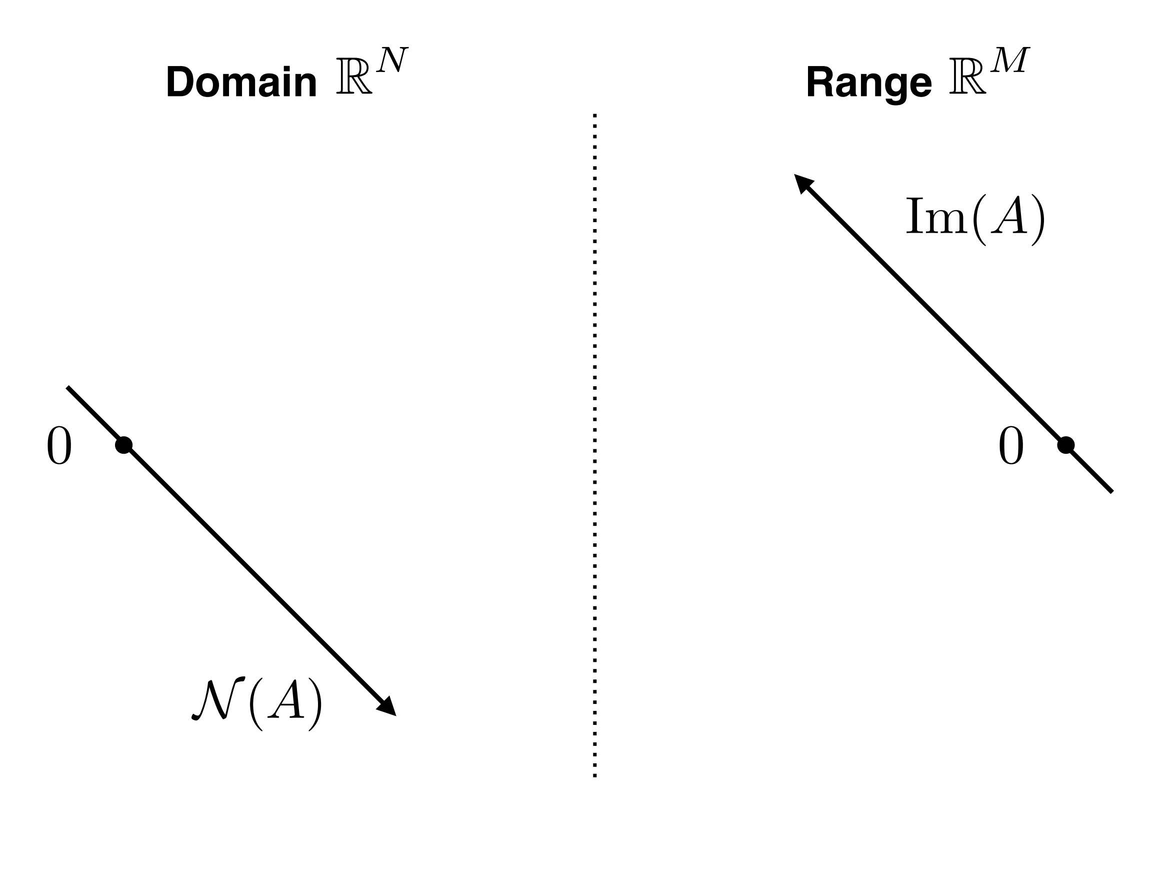
Note that 0 is always in the image. It's the output when 0 is the input.

Null Space of a Matrix

The Null Space of a Matrix is the Pre-image of 0.

The Null Space is also called the Kernel.

Note that 0 is always in the pre-image. Its output is 0 .



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Consider the Transpose

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_N \\ | & | & & | \end{bmatrix} \quad A^T = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_N^T & — \end{bmatrix}$$

The rows of A^T **are the columns of** A**.**

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Consider any vector in the null space of A^T.

$$A^T v = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_N^T & — \end{bmatrix} v = \begin{bmatrix} a_1^T v \\ a_2^T v \\ \vdots \\ a_N^T v \end{bmatrix} = 0$$

v **must be perpendicular to the columns of** A **.**

So every vector in the null space is perpendicular to every vector in the Image.

$$\mathcal{N}(A^T) \perp \mathrm{Im}(A)$$

Theorem

Any vector in $\mathbb{R}^M$ is either in the image of A or the null space of A^T.

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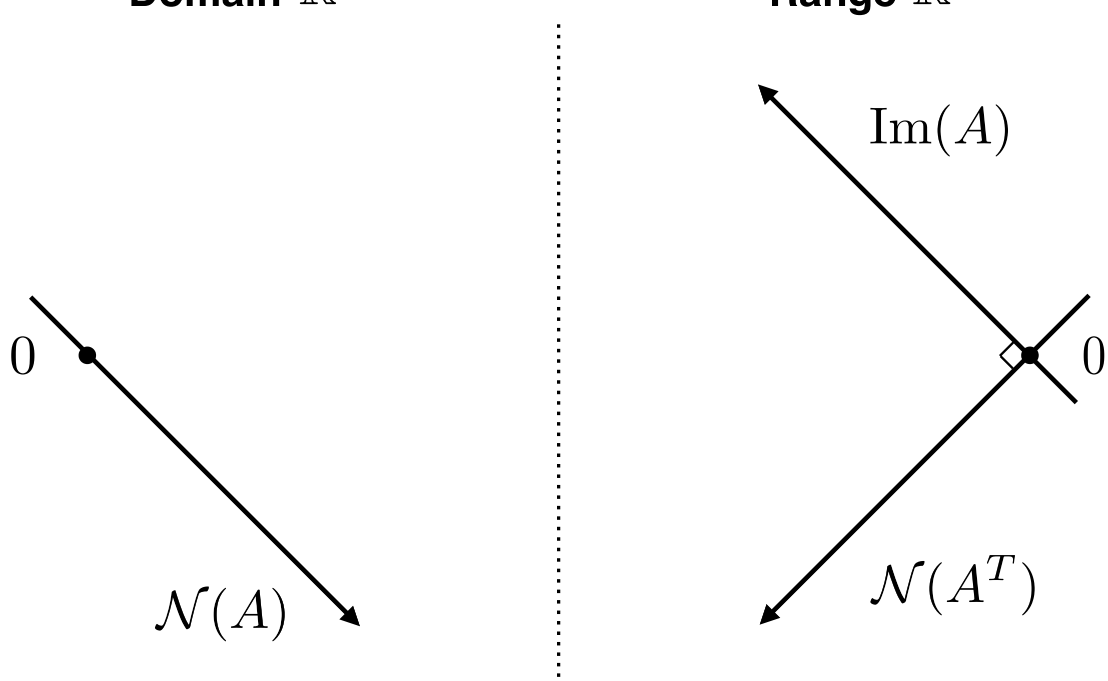
Domain $\mathbb{R}^N$ **Range** $\mathbb{R}^M$

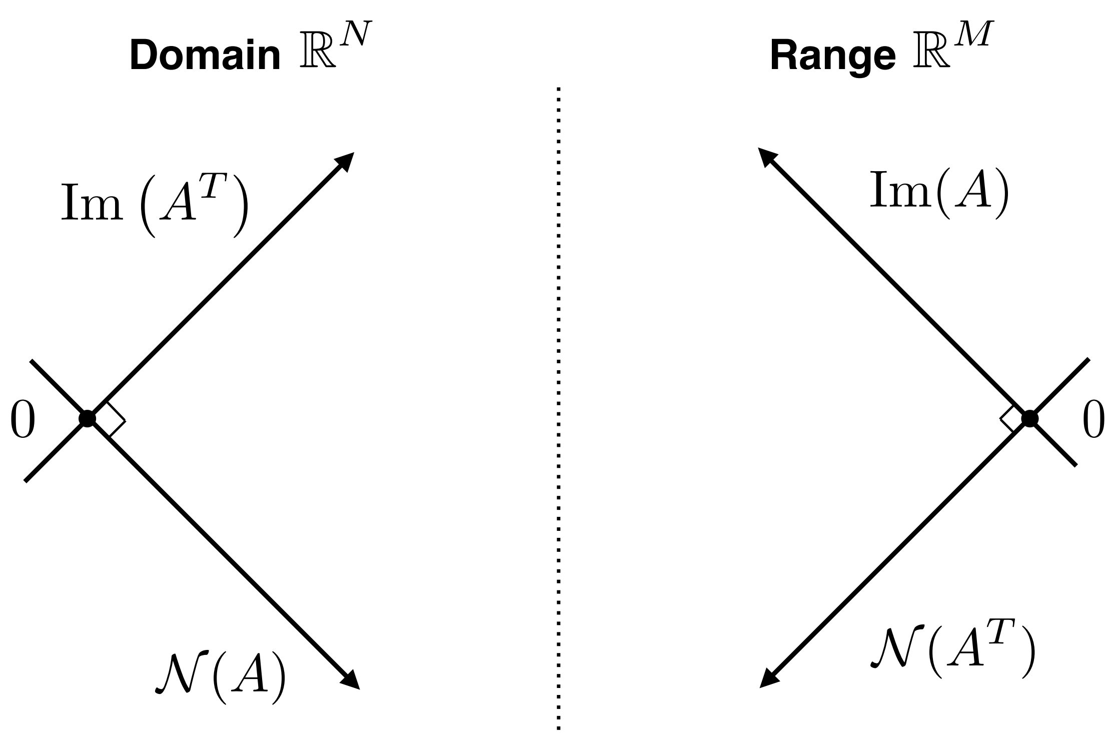


So every vector in the null space is perpendicular to every vector in the Image.

$$\mathcal{N}(A^T) \perp \mathrm{Im}(A)$$

We could make the same argument in the other direction.

$$\mathcal{N}(A) \perp \mathrm{Im}\left(A^T\right)$$



These are the four subspaces.

Matrix-Matrix Multiplication

$$U \underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_N \\ | & | & & | \end{bmatrix}}_{V} = \begin{bmatrix} | & | & & | \\ U\,v_1 & U\,v_2 & \cdots & U\,v_N \\ | & | & & | \end{bmatrix}$$

Each column of the output is the result of the matrix *U* times the corresponding column of the matrix *V*.

Matrix-Matrix Multiplication

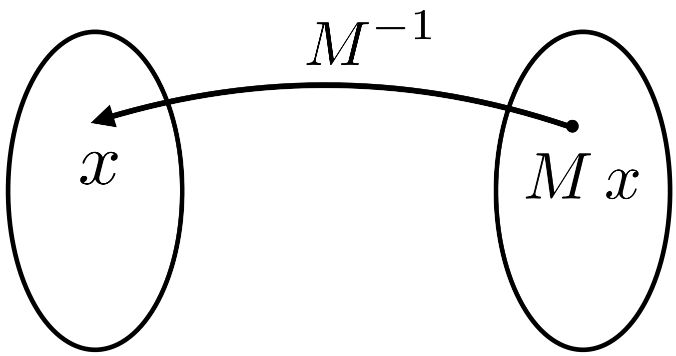
$$\begin{bmatrix} — & r_{u,1}^T & — \\ — & r_{u,2}^T & — \\ & \vdots & \\ — & r_{u,M}^T & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ c_{v,1} & c_{v,2} & \cdots & c_{v,N} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} r_{u,1}^T c_{v,1} & r_{u,1}^T c_{v,2} & \cdots & r_{u,1}^T c_{v,N} \\ r_{u,2}^T c_{v,1} & r_{u,2}^T c_{v,2} & \cdots & r_{u,2}^T c_{v,N} \\ \vdots & & \ddots & \vdots \\ r_{u,M}^T c_{v,1} & r_{u,M}^T c_{v,2} & \cdots & r_{u,M}^T c_{v,N} \end{bmatrix}$$

Each element of the output is a dot product of the rows of the first matrix with the columns of the second.

Matrix Inverse

For some matrices, there exists an inverse matrix such that

$$M^{-1}M = I$$



Note: it's a very special thing for a matrix to be invertible.

Theorem

Only square matrices can be invertible.

Block Matrix

A matrix where each element is a matrix.

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & & A_{2N} \\ \vdots & & \ddots & \vdots \\ A_{M1} & A_{M2} & & A_{MN} \end{bmatrix}$$

Here, each A_{ij} **is a matrix.**

Block Matrix Multiplication

Block Matrix Multiplication is just like the dot product matrix multiplication.

$$\begin{bmatrix} — & r_{A,1}^T & — \\ — & r_{A,2}^T & — \\ & \vdots & \\ — & r_{A,M}^T & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ c_{B,1} & c_{B,2} & \cdots & c_{B,N} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} r_{A,1}^T c_{B,1} & r_{A,1}^T c_{B,2} & \cdots & r_{A,1}^T c_{B,N} \\ r_{A,2}^T c_{B,1} & r_{A,2}^T c_{B,2} & \cdots & r_{A,2}^T c_{B,N} \\ \vdots & & \ddots & \vdots \\ r_{A,M}^T c_{B,1} & r_{A,M}^T c_{B,2} & \cdots & r_{A,M}^T c_{B,N} \end{bmatrix}$$