Vectors

Math Lecture 2

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Vectors

A vector is an ordered finite list of numbers together with pointwise addition and scalar multiplication.

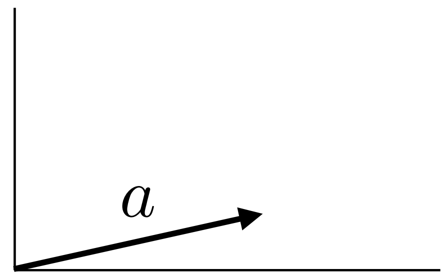
Example: $\begin{bmatrix} -1.1 \\ 0.0 \\ 3.6 \\ -7.2 \end{bmatrix}$ $(-1.1, 0.0, 3.6, -7.2)$

Example: 0

All the elements are 0.
The length is understood from context.

Drawing Vectors in 2D

$$a = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$



Vector Addition

Two vectors of the same size can be added together by adding corresponding components.

Example:

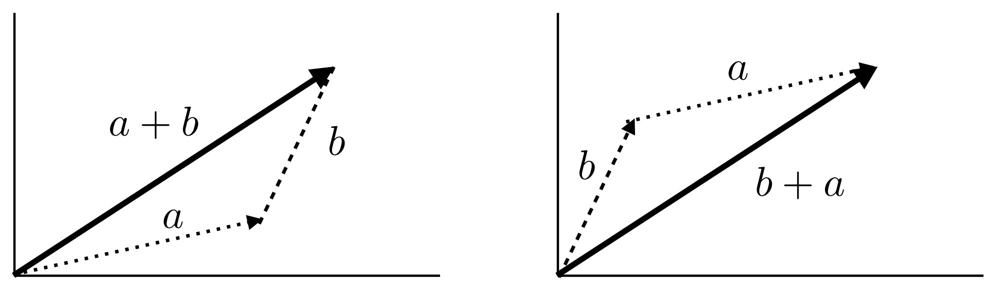
$$\begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 \\ 9 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

Geometric Interpretation

Vectors add tip-to-tail.



Scalar Multiplication

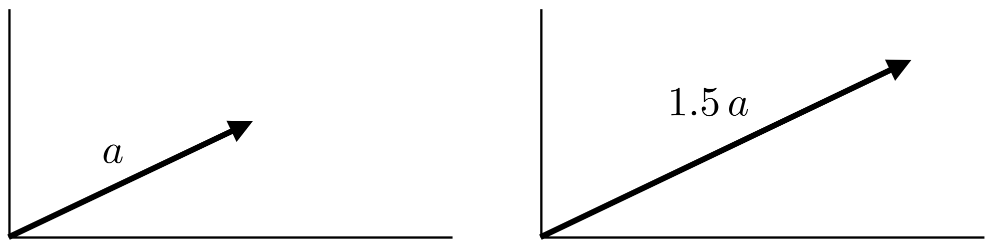
Every element of the vector is multiplied by the scalar (i.e. number)

Example:

$$(-2)\begin{bmatrix}1\\9\\-6\end{bmatrix} = \begin{bmatrix}-2\\-18\\12\end{bmatrix}$$

Geometric Interpretation

Vector is scaled by scalar multiplication.

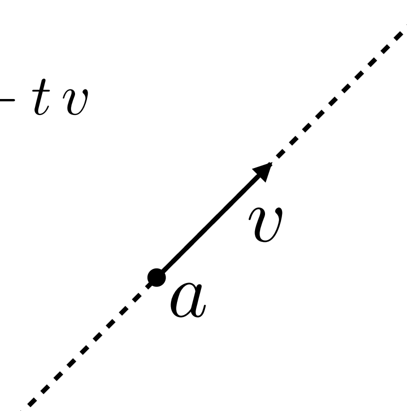


Parametric Equation of a Line

Suppose that a is a point on the line and v is a vector parallel to the line. The line can be represented as

$$f(t) = a + t\,v$$

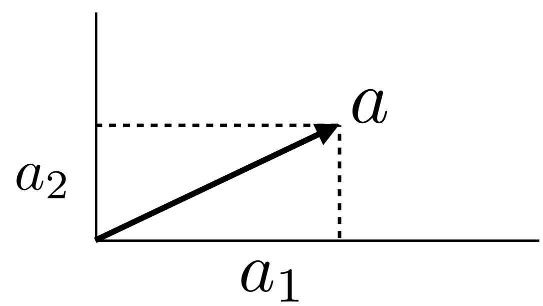
where t is any real number.



Length of a Vector

The length of a vector a **, denoted by** $\|a\|_2$ **, is**

$$\|a\|_2 = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$



Dot Product

If a **and** b **are vectors then**

$$a \cdot b = a^T\, b = a_1\, b_1 + a_2\, b_2 + \cdots + a_n\, b_n$$

Cross Product

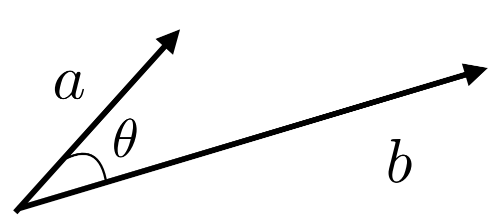
If a **and** b **are vectors in** $\mathbb{R}^3$

$$a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + b_1 a_3 \\ a_1 b_2 - b_1 a_2 \end{bmatrix}$$

Angle Between Two Vectors

Let θ **denote the angle between vectors** a **and** b **.**

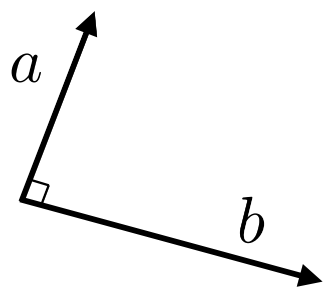
$$\theta = \arccos\left(\frac{a^T b}{\|a\|_2 \, \|b\|_2}\right)$$



Perpendicular Vectors

Two vectors a and b are perpendicular if and only if

$$a \cdot b = 0$$



Dot Product Properties

The angle between two vectors *a,b* is acute if and only if

$$a \cdot b > 0$$

The angle between two vectors *a,b* is obtuse if and only if

$$a \cdot b < 0$$

Cross Product Properties

Define c to be $c = a \times b$.

Then c is perpendicular to a and c is perpendicular to b

$$c \perp a \qquad\qquad c \perp b$$

Cauchy-Schwarz Inequality

Bounds the magnitude of the inner product between two vectors

$$|a^T\, b| \leq \|a\|_2\, \|b\|_2$$

Linear Combination

Suppose $a_1, a_2, \ldots, a_n$ **are vectors of the same size.**

A linear combination of these vectors is an expression of the form

$$\beta_1\, a_1 + \beta_2\, a_2 + \cdots + \beta_n\, a_n$$

where $\beta_1, \beta_2, \ldots, \beta_n$ **are numbers.**

Linearly Independent

A set of vectors $a_1, a_2, \ldots, a_n$ **is Linearly Independent means the only solution to**

$$c_1\, a_1 + c_2\, a_2 + \cdots + c_n\, a_n = 0$$

is $c_1 = c_2 = \cdots = c_n = 0$

Span

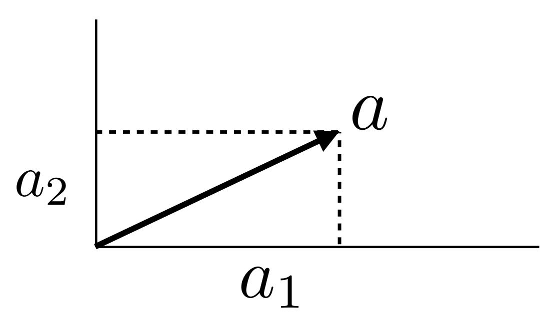
Suppose $a_1, a_2, \ldots, a_n$ are vectors of the same size.

The span of $\{a_1, a_2, \ldots, a_n\}$ is the set of *all* linear combinations of the vectors in the set.

L2 Norm

The L2 norm of a vector a , denoted by $\|a\|_2$, is

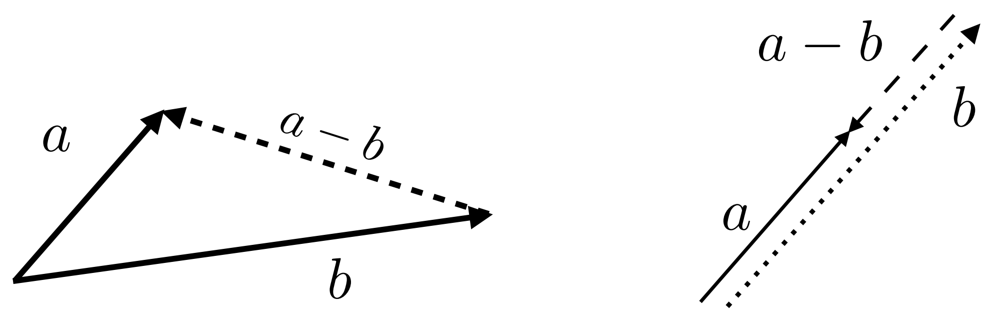
$$\|a\|_2 = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$



Metric of Similarity - L2 Norm

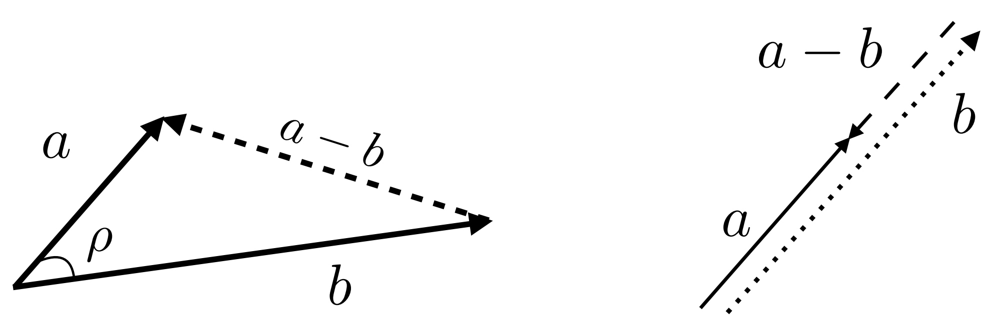
If the L2 norm = 0, the vectors are identical

$$\|a - b\|_2$$



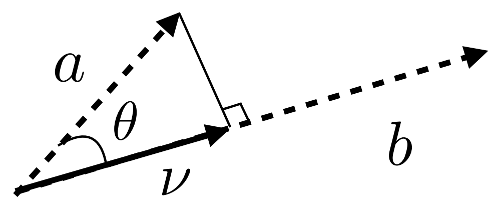
Metric of Similarity - Pearson Correlation Coefficient

$$\rho = \frac{a^T\, b}{\|a\|_2\, \|b\|_2}$$



Vectors are considered identical

Vector Projection



ν **is called the projection of vector** a **onto** b **.**

$$\nu = \mathrm{proj}_b a = \frac{a^T b}{\|b\|_2^2} b$$