Optimization

Math Lecture 4

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Main Goal

Find x such that $Ax \approx b$.

Three possibilities

- **There doesn't exist any x that satisfies**
- **There exists exactly 1 x that satisfies**
- **There exists infinitely many x that satisfies**

One Solution

If there exists a solution, then we seek the x that satisfies $A\,x = b$.

Infinite Solutions

If there exist infinite solutions, then we seek the smallest x that satisfies

$$A\,x = b$$

No Solutions

If there doesn't exist any *x* such that $A\,x = b$ then we seek the smallest *x* such that

$$A\,x \approx b$$

Formulating the Problem

What exactly do we mean by $A\,x \approx b$?

We can mean that we want any one of the following to be small:

$$\|A\,x - b\|_2 \qquad \|A\,x - b\|_1 \qquad \|A\,x - b\|_*$$

$$\sum A\,x - b \qquad \prod A\,x - b \qquad \|A\,x - b\|_\infty$$

$$\|A\,x - b\|_0 \qquad \max\left(A\,x - b\right) \qquad \min\left(A\,x - b\right)$$

Our Problem

In this class, we will focus on converting problems into the following formulation:

$$\text{minimize} \quad \|A\,x - b\|_2$$

Our computers can solve this problem!

Mathematical computer programs have a Least Squares algorithm built in

In python

```
x = numpy.linalg.lstsq( A, b )
```

In Matlab

```
x = A \ b;
```

As Engineers

We are given a problem in English

Ex: Fill in the missing parts of a damaged picture

We translate this problem into the form

$$\text{minimize} \quad \|A\,x - b\|_2$$

We find the solution using computer programs that we make

Regularization

We can account for additional information through Regularization.

$$\text{minimize} \quad \|Ax - b\|_2 + \gamma\, R(x)$$

R **is the regularization function.**

Tikhonov Regularization

$$\text{minimize} \quad \|A\,x - b\|_2^2 + \gamma\,\|\Gamma\,x\|_2^2$$

Γ **is called the Tikhonov matrix**

γ **is the regularization parameter.**
Trades off importance of data matching and regularization

Tikhonov Regularization

$$\text{minimize} \quad \|A\,x - b\|_2^2 + \gamma\,\|\Gamma\,x\|_2^2$$

Example Tikhonov matrices:

D **-** *Dx* **is a vector of all the horizontal and vertical differences. Used if** *x* **is expected to be smooth**

I **- identity matrix is used when x is expected to be small**

Tikhonov Regularization

$$\text{minimize} \quad \|A\,x - b\|_2^2 + \gamma\,\|\Gamma\,x\|_2^2$$

Can be combined into a minimization of a single term.

You will do this for homework.