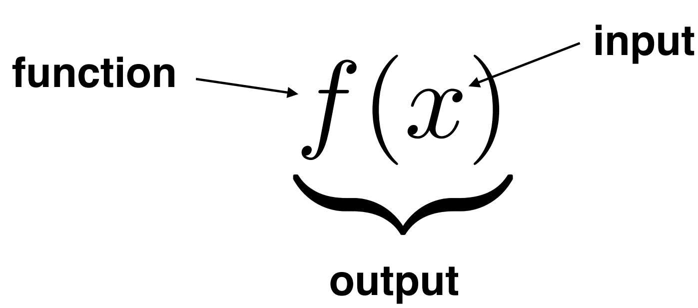
Functions

Math Lecture 6

Nicholas Dwork

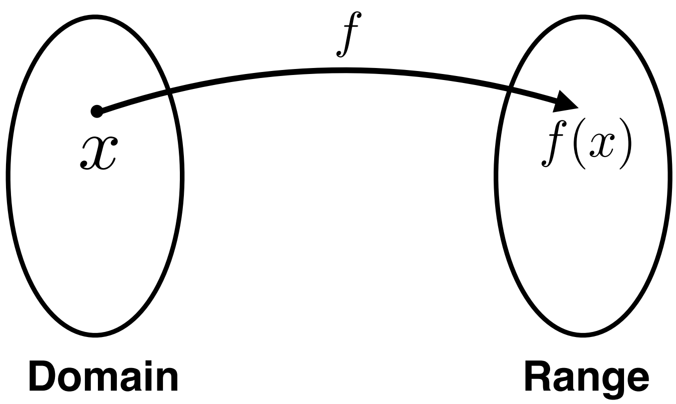
Functions

A function is a mathematical machine
You input something
You get something out



As long as you input the same thing, you'll always get the same thing out.

Consider a function f . What is the precise definition of a function?

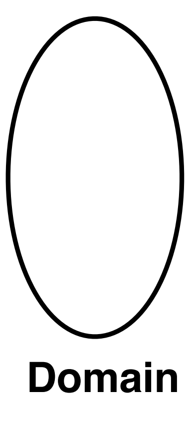


As the picture above indicates, a function has three parts. We define a function as an ordered triple.

Domain

A function is an ordered triple.

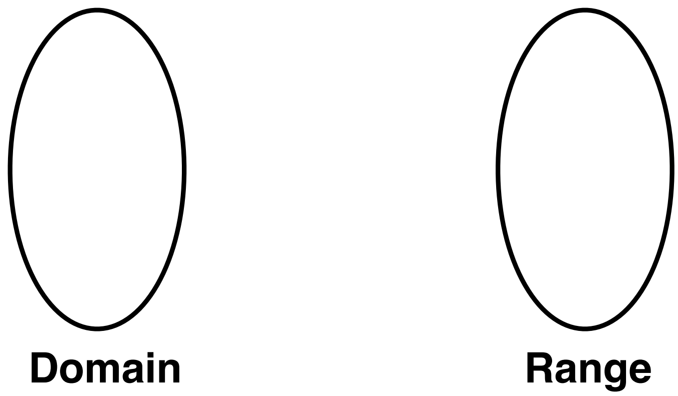
The first element is a set called the Domain.



Range

A function is an ordered triple.

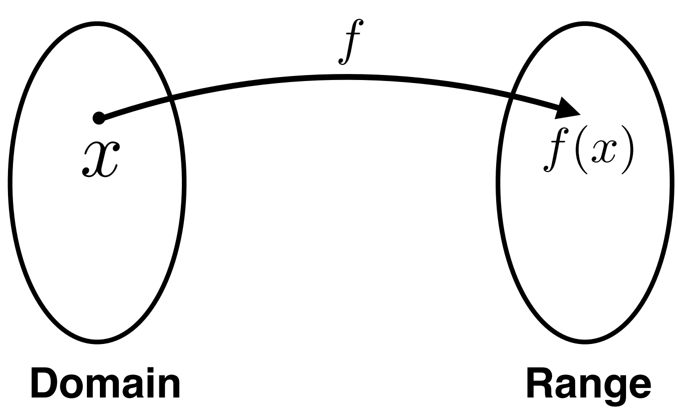
The second element is a set called the Range.



Set of Ordered Pairs

A function is an ordered triple.

The third element is a set of ordered pairs that matches elements in the domain with elements in the Range.



Set of Ordered Pairs

$$(x, f(x))$$

For each element in the Domain there is an ordered pair.

The first element in the pair is the Domain element. This is often called the “input”.

The second element of the ordered pair is an element of the Range. Its often called the “output” for that input.

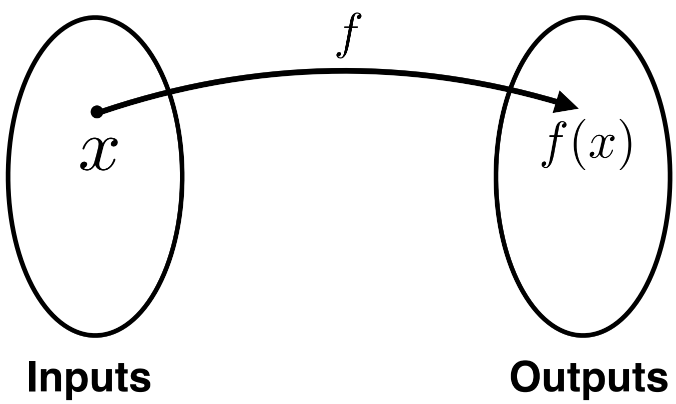
A Function

Domain Range

$$f = (D, R, S)$$

$$S = \{(x, f(x)) : x \in D\}$$

A function maps inputs to outputs. It converts x to $f(x)$.



Function Notation

$$f : D \to R$$

This is followed by "such that" and then a rule specifying $f(x)$.

Functions - Example 1

$f : \mathbb{R} \rightarrow \mathbb{R}$ **such that** $f(x) = x$

Functions - Example 2

Domain **Range**

({a,b,c,d}, {0,1,2,3,4,5,6,7,8},
{ (a,0), (b,0), (c,8), (d,6) })

S

This is a completely valid function!

Functions - Example 3

$f : \mathbb{R} \to \{0, 1\}$ **such that**

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

Example Function

The exponential function: $\exp$

$\exp(-1) = 0.3679$

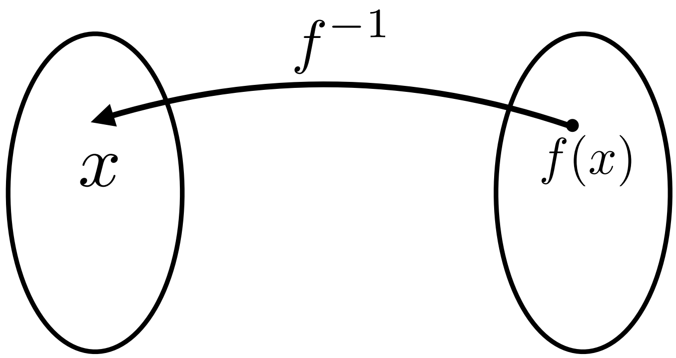
$\exp(0) = 1$

$\exp(1) = 2.7183$

$\exp(1.2) = 3.3201$

Inverse Function

The inverse function converts all $f(x)$ in Outputs back to x in Inputs.



Note: Not every function has an inverse function. An invertible function is a very special thing.

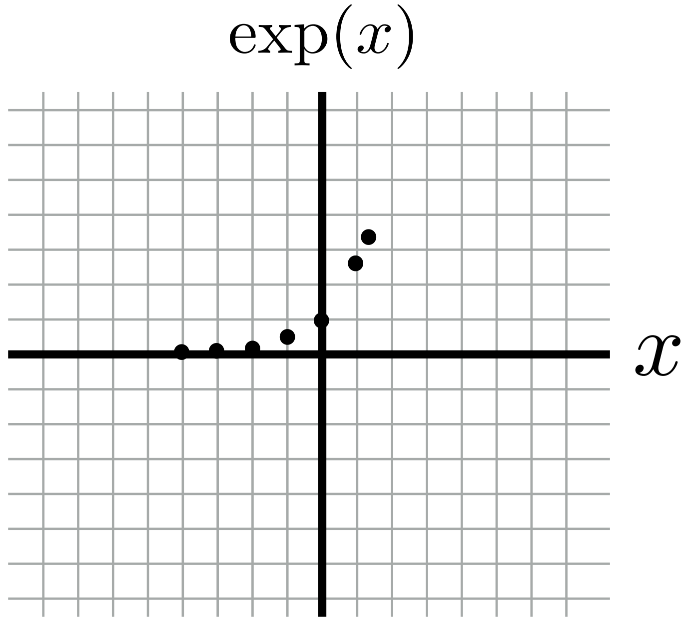
Example Inverse Function

The inverse of the $\exp$ function is the $\log$ function.

$\exp(-1) = 0.3679$	$\log(0.3679) = -1$
$\exp(0) = 1$	$\log(1) = 0$
$\exp(1) = 2.7183$	$\log(2.7183) = 1$
$\exp(1.2) = 3.3201$	$\log(3.3201) = 1.2$

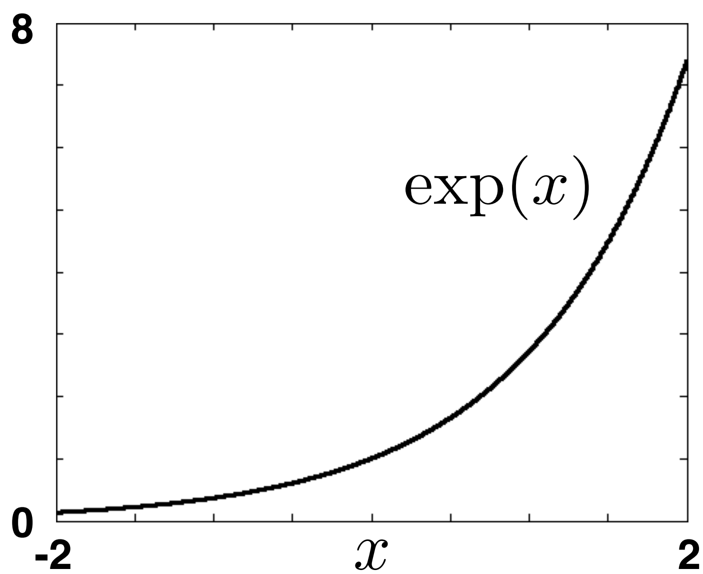
Graphing a Function

Showing the points of a function in a Euclidean Plane



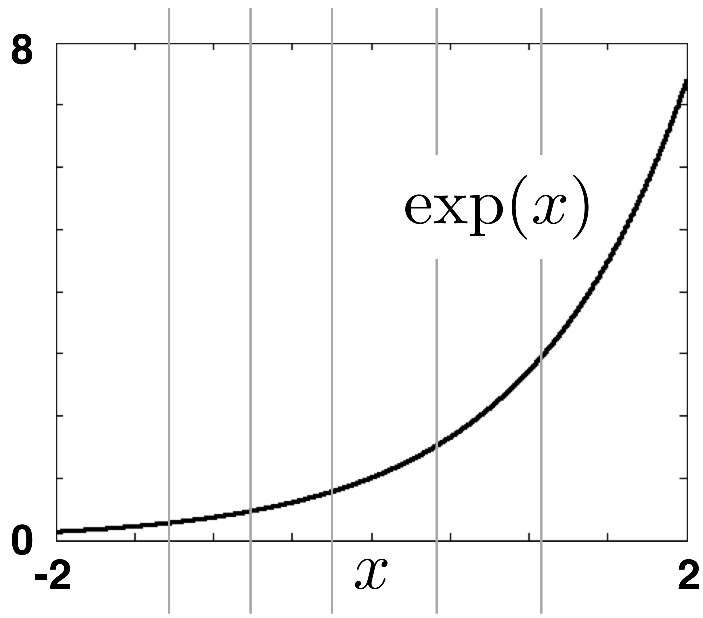
Graphing a Function

Showing all the points of a function in a Euclidean Plane



Vertical Line Test

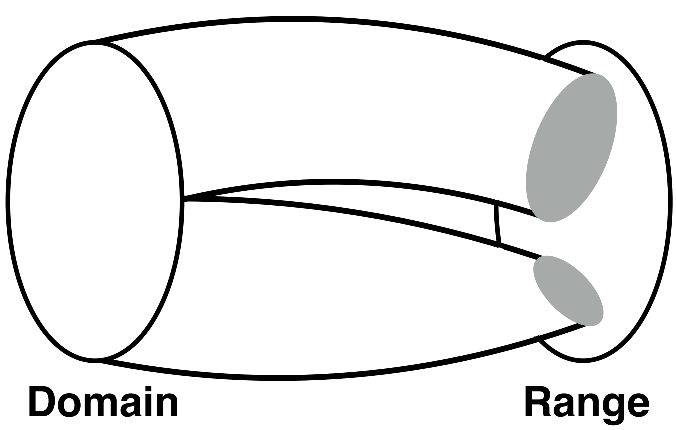
If you draw a vertical line through the graph of a function, it will intersect at most one point.



Image

The set of all attainable outputs is called the Image.

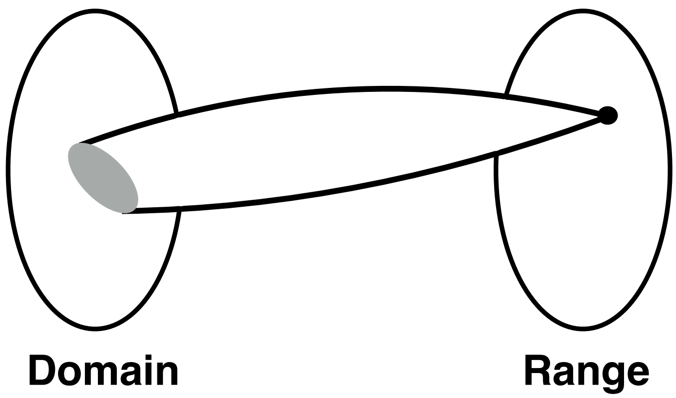
The Image of a function is a subset of the Range.



Pre-image

The set of all inputs that map to a value is called the pre-image of that value for the function.

The Pre-image of a value is a subset of the Domain.



Equation of a Line

One way to represent a line is with a function of the form

$$f(x) = m\,x + b$$

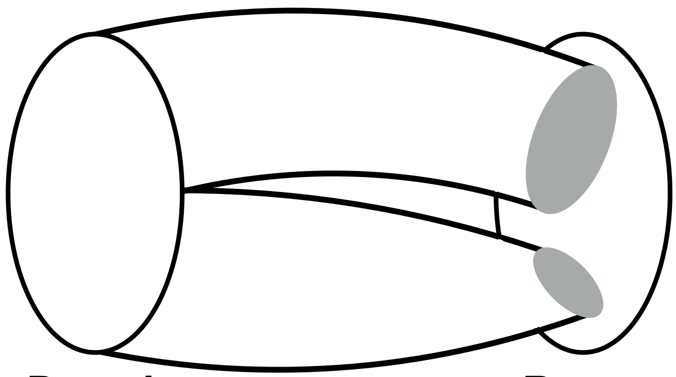
m **is called the "slope" of the line.**
b **is called the "vertical intercept" of the line.**

We can't represent vertical lines this way.

Image of a Matrix

The set of all $M\,x$ is called the Image of M.

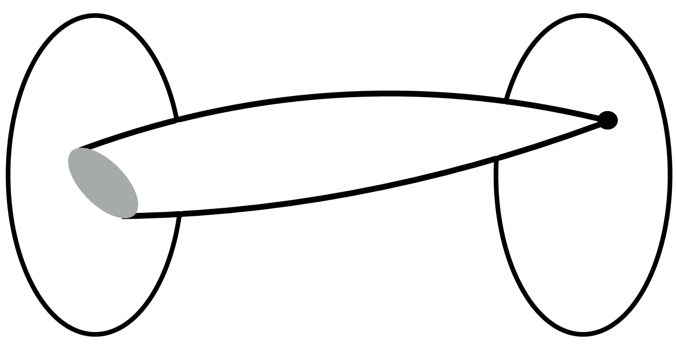
The Image of a matrix is a subset of the Range.



Pre-image of a Matrix

The set of all inputs that map to a value is called the pre-image of that value for the Matrix.

The Pre-image of a value is a subset of the Domain.



$$\mathrm{Im}(M) = \left\{ Mx \text{ for all } x \in \mathbb{R}^M \right\}$$

$$\text{Pre-image}(y) = \{x \text{ such that } Mx = y\}$$

Question:
What is the Image of Matrix A?

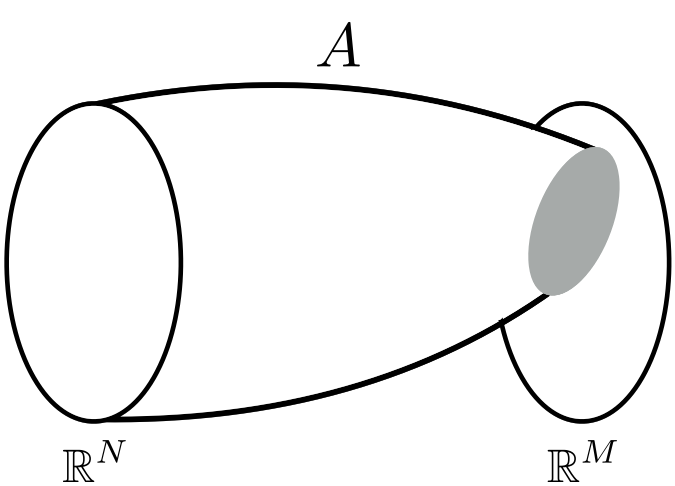


Image of a Matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} v = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{M1} \end{bmatrix} v_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{M2} \end{bmatrix} v_2 + \cdots + \begin{bmatrix} a_{1N} \\ a_{2N} \\ \vdots \\ a_{MN} \end{bmatrix} v_N$$

From the definition of Matrix-Vector multiplication, we see that the Image of a Matrix is the span of its columns.

Another name for the Image of a Matrix is "column space".

Image of a Matrix

$$\mathrm{Im}(A) = \mathrm{span}(c_1, c_2, \cdots, c_N)$$

where c_i is the i^{th} column of a matrix.

Note that 0 is always in the image. It's the output when 0 is the input.

Null Space of a Matrix

The Null Space of a Matrix is the Pre-image of 0.

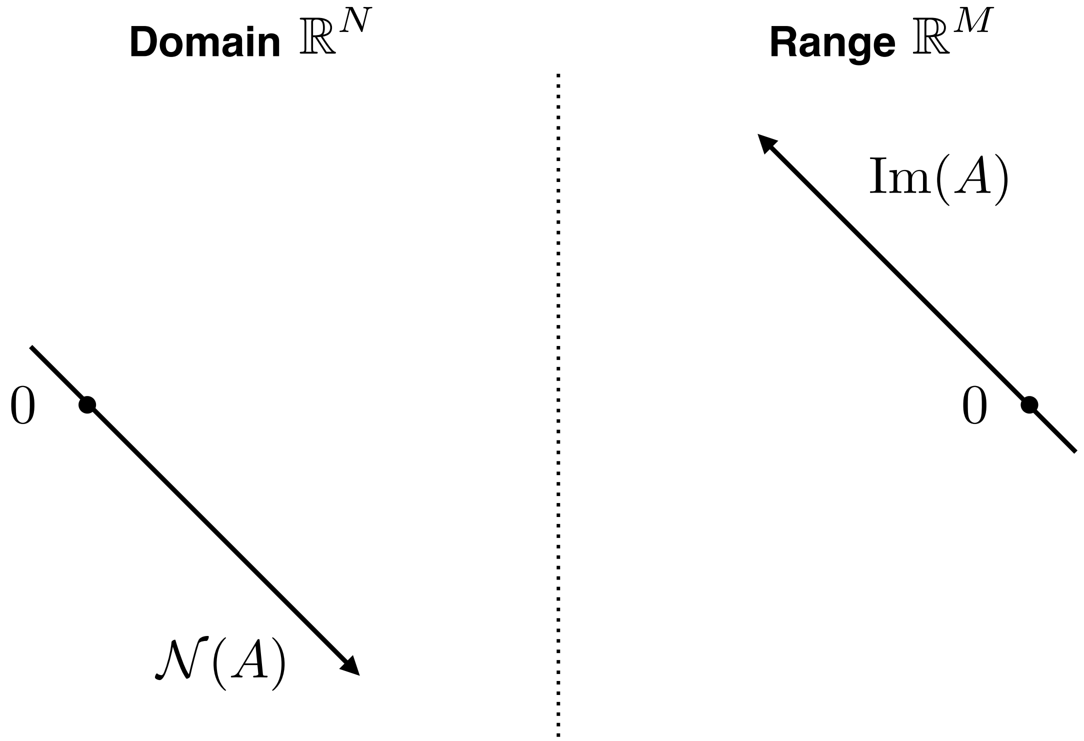
The Null Space is also called the Kernel.

Note that 0 is always in the pre-image. Its output is 0 .

Null Space of a Matrix

The Rank of a matrix is the dimension of its image

The Nullity of a matrix is the dimension of its null space



Consider the Transpose

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_N \\ | & | & & | \end{bmatrix} \quad A^T = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_N^T & — \end{bmatrix}$$

The rows of A^T are the columns of A.

Consider any vector in the null space of A^T.

$$A^T v = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_N^T & — \end{bmatrix} v = \begin{bmatrix} a_1^T v \\ a_2^T v \\ \vdots \\ a_N^T v \end{bmatrix} = 0$$

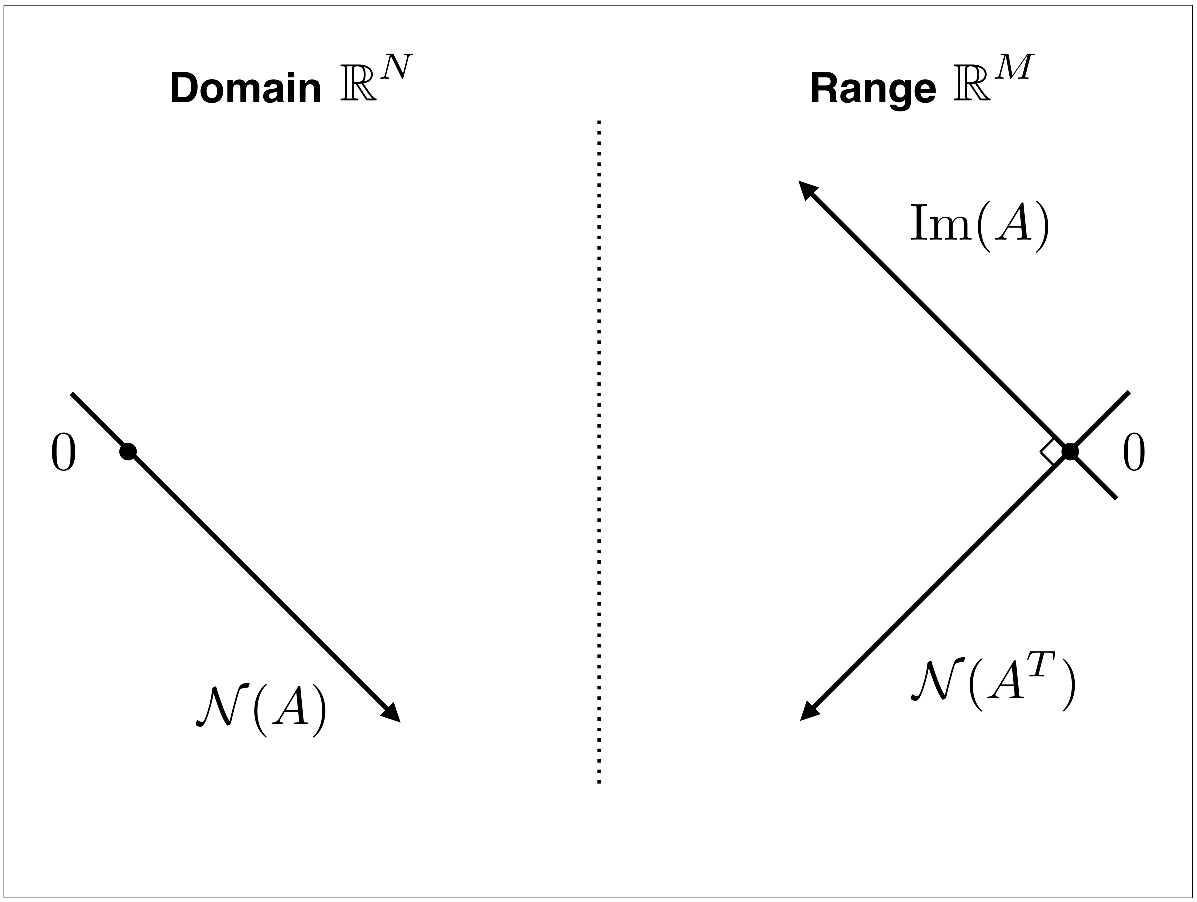
v **must be perpendicular to the columns of** A **.**

So every vector in the null space is perpendicular to every vector in the Image.

$$\mathcal{N}(A^T) \perp \mathrm{Im}(A)$$

Theorem

Any vector in $\mathbb{R}^M$ is either in the image of A or the null space of A^T.

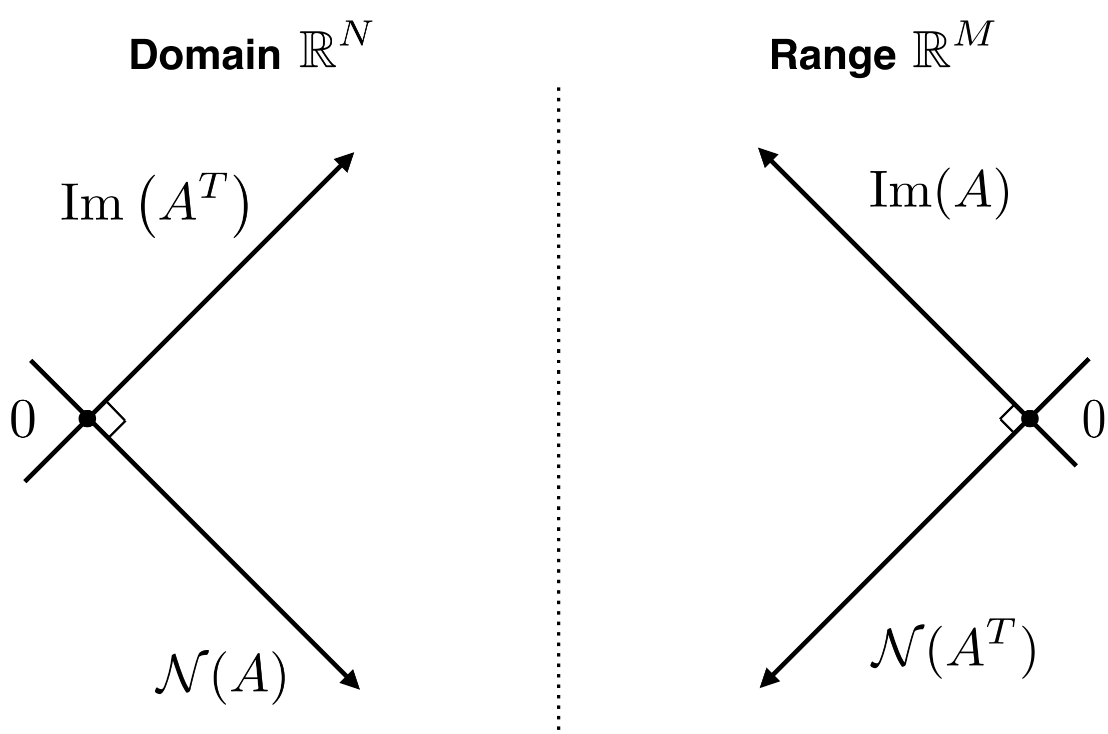


So every vector in the null space is perpendicular to every vector in the Image.

$$\mathcal{N}(A^T) \perp \mathrm{Im}(A)$$

We could make the same argument in the other direction.

$$\mathcal{N}(A) \perp \mathrm{Im}\left(A^T\right)$$

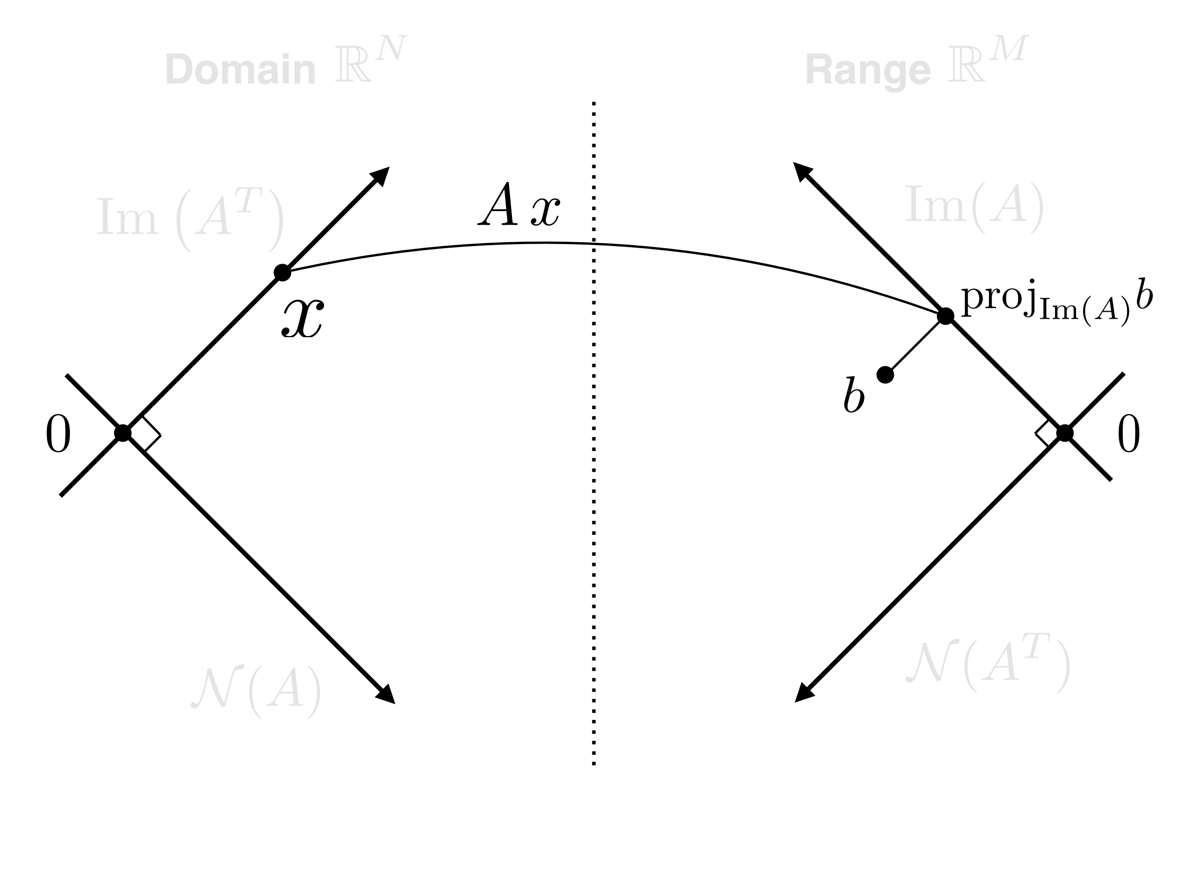


These are the four subspaces.

Pseudo-Inverse

The Pseudo-Inverse of A is the matrix $A^\dagger$ such that $A^\dagger b$ is the solution to the following optimization problem (for any b).

$$\begin{aligned} \text{minimize} \quad & \|x\|_2 \\ \text{subject to} \quad & Ax = \text{proj}_{\text{Im}(A)} b \end{aligned}$$



Most common situation is that there are no solutions and A is tall and skinny with linearly independent columns

$$A\,x = b$$

Left multiply by A^T

$$A^T A\,x = A^T b$$

The expression above is called the "Normal Equations"

$$A^T A\,x = A^T b$$

Since A **has linearly independent columns,** $A^T A$ **is invertible.**

$$x = \left(A^T A\right)^{-1} A^T b$$

Now we see how to determine the pseudo-inverse of A **for this situation**

$$A^\dagger = \left(A^T A\right)^{-1} A^T$$