The Four Subspaces
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1 Preliminaries

Let \( A \in \mathbb{R}^{m \times n} \).

**Def:** The Null Space of \( A \) (also called the kernel of \( A \)), denoted \( N(A) \), is the set \( \{ x : Ax = 0 \} \).

**Def:** The Image of \( A \) (also called the column space of \( A \)) denoted \( \text{Im}(A) \), is the set \( \{ Ax : x \in \mathbb{R}^n \} \).

**Def:** The dimension of a vector space is the number of elements in any basis of the vector space.

**Def:** The Nullity of \( A \), denoted \( N(A) \), is the dimension of \( N(A) \).

**Def:** The Rank of \( A \), denoted \( \text{rank}(A) \), is the dimension of \( \text{Im}(A) \).

**Def:** A subset of a vector space that is itself a vector space is called a subspace.

**Def:** The sum of two vector spaces \( V, W \), denoted \( V + W \), is the set \( \{ v + w : v \in V, w \in W \} \).

**Def:** A vector space \( V \) is called the direct sum of two vector spaces \( W_1, W_2 \), denoted by \( V = W_1 \oplus W_2 \), if \( W_1, W_2 \) are subspaces of \( V \) such that \( W_1 \cap W_2 = \{ 0 \} \) and \( W_1 + W_2 = V \).

**Def:** The inner product of \( x, y \) is denoted by \( \langle x, y \rangle \).

**Def:** Two vectors \( x, y \) are orthogonal means \( \langle x, y \rangle = 0 \).

**Def:** The dot product of two vectors \( x, y \in \mathbb{R}^n \) is \( x \cdot y = x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \).

**Def:** The orthogonal complement of a vector space \( V \), denoted \( V^\perp \), is the set \( \{ w : \langle v, w \rangle = 0 \ \forall \ v \in V \} \).

2 Euclidean Vector Spaces

We will consider a matrix \( A \in \mathbb{R}^{m \times n} \). Under multiplication, \( A : \mathbb{R}^n \to \mathbb{R}^m \). To aid us in our understanding of the four subspaces, we will draw both the domain and range of \( A \) side by side. The dots in the picture are the origins, denoted by 0.

![Diagram](image)

The set \( N(A) \) is a subspace of \( \mathbb{R}^n \). Any vector \( x \in N(A) \) maps to 0 under matrix multiplication with \( A \). Here we draw \( N(A) \) as a line through the origin. Recall that any subspace of a vector space must include the origin. Any point \( x \in N(A) \) is mapped to the origin of \( \mathbb{R}^m \) by multiplication with \( A \).

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\(^1\)It is also often called the Range of \( A \), but I find that name confusing since it conflicts with the definition of the range of a function, and I don’t recommend it.
The set $\text{Im}(A)$ is a subspace of $\mathbb{R}^m$. Any vector $x \in \mathbb{R}^n$ maps to $\text{Im}(A)$. Any vector $x \in \mathbb{R}^n - N(A)$ maps to a nonzero vector in $\text{Im}(A)$, as shown below.

Let us consider the matrix $A^T \in \mathbb{R}^{n \times m}$. Similar to our discussion of $A$, $N(A^T)$ is a subspace of $\mathbb{R}^m$ and $\text{Im}(A^T)$ is a subspace of $\mathbb{R}^n$.

The four sets shown in the figure above, $\text{Im}(A), \text{Im}(A^T), N(A), N(A^T)$ are called the four subspaces. Denote the $i$th column of $A$ by $a_i$. Then

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix}.$$

Consider a vector $x \in N(A^T)$. This means that $A^T x = 0$. Written out more explicitly,

$$\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_n^T x \end{bmatrix} = 0.$$

This shows that $x$ is orthogonal to all the columns of $A$. Since $\text{Im}(A)$ equals the span of the columns of $A$, $x$ is orthogonal to $\text{Im}(A)$. And since $x$ was any vector in $N(A^T)$, $N(A^T) \perp \text{Im}(A)$. Similarly, $N(A) \perp \text{Im}(A^T)$.  


From the above, we see that \( N(A) \oplus Im(A^T) = \mathbb{R}^n \) and \( N(A^T) \oplus Im(A) = \mathbb{R}^m \).

The above picture summarizes some very powerful knowledge in Linear Algebra. As an example of this power, we'll now use our new understanding to prove two very famous theorems.

**The Rank-Nullity Theorem:** \( rank(A) + \overline{N}(A) = n \).

**Proof:** Let \( \alpha \) be a basis for \( N(A) \) with dimension \( a \) where \( a \leq n \). We can find a basis \( \beta \) for \( Im(A^T) \) of dimension \( n - a \). \( dim(A(\beta)) = dim(\beta) \), and \( dim(A(\beta)) = dim(Im(A)) = rank(A) \). Therefore, \( rank(A) = dim(\beta) = n - \overline{N}(A) \).

**Theorem:** \( rank(A) = rank(A^T) \).

**Proof:** Since \( Im(A^T) \oplus N(A) = \mathbb{R}^n \), \( rank(A^T) + \overline{N}(A) = n \). From the last theorem, we know \( rank(A) + \overline{N}(A) = n \). Subtracting one equation from the other yields \( rank(A) - rank(A^T) = 0 \).

The previous theorem shows us that \( A \) (when viewed as a mapping) is one-to-one when restricted to \( Im(A^T) \). It's like \( Im(A^T) \) is the “true” domain of \( A \), and the vectors in \( N(A) \) are just a distraction.

### 3 Inner Product Spaces

So far we've been working in Euclidean vector spaces. Now we'll work in the more general inner product spaces. We'll start by presenting more relevant definitions.

**Def:** A function \( f \) is linear means \( f(x + y) = f(x) + f(y) \) and \( f(kx) = kf(x) \) for any scalar \( k \).

**Def:** A function \( T \) is a linear transformation means \( T \) is a linear function and it maps one vector space onto another.

**Def:** The adjoint of a linear transformation \( T: V \to W \), denoted \( T^* \), is the linear transformation that satisfies \( \langle Tx, y \rangle = \langle x, T^*y \rangle \) for all \( x \in V, y \in W \).

**Def:** A function \( T \) is a linear operator means \( T \) is a linear transformation that maps a vector space onto itself.

**Def:** The null space of \( T: V \to W \), denoted \( N(T) \), is the set \( x \in V : T(x) = 0 \).

**Def:** The image of \( T: V \to W \), denoted \( Im(T) \), is the set \( \{ T(x) : x \in V \} \).

**Def:** An inner product space is a vector space together with an inner product.

Let \( T: V \to W \) such that \( T \) is a linear transformation and \( V, W \) are finite dimensional inner product spaces.

**Theorem:** \( Im(T) \perp N(T^*) \).

**Proof:** Let \( y \in N(T^*) \). Then \( T^*(y) = 0 \), which implies \( \langle x, T^*y \rangle = 0 \) for any \( x \in V \). But \( \langle x, T^*y \rangle = \langle T(x), y \rangle \), so \( \langle T(x), y \rangle = 0 \). This shows that \( y \) is perpendicular to \( Im(T) \). And, since \( y \) was any vector in \( N(T^*) \), the proof is complete.

From here, we gain an understanding like before that is summarized in the figure below.